DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ANALYSIS

January 2006

Instructions

1. Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for a correct application even though you do not give a correct proof.

2. Use a different sheet (or different set of sheets) for each question. Write the problem number and your code number (not your name) on the top of every sheet.

3. Keep scratch work on separate sheets.

4. Unless otherwise stated, you may appeal to a "well-known theorem" in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.
1. Let $f \in L^1[0, \infty)$, $f \geq 0$. Show that
\[
\lim_{n \to \infty} \frac{1}{n} \int_0^n x f(x) \, dx = 0.
\]

2. Let $u$ be (real valued) harmonic in the annulus $A = \{1 < |z| < 2\}$. Prove there is $f(z)$ analytic on $A$ and real $b$ so that
\[
u = b \log |z| + \Re\{f(z)\}
\]

3. Let $k \in L^1(R), k \geq 0, \int_R k = 1$.
   (a) For each $\delta > 0$, prove that
   \[
   \lim_{n \to \infty} \int_{|x| \geq \delta} nk(nx) \, dx = 0.
   \]
   (b) If $g$ is real-valued, bounded, and continuous on $R$, show that
   \[
   \lim_{n \to \infty} \int_R nk(nx)g(x) \, dx = g(0).
   \]

4. (a) Given a constant $K$ so that a sequence of complex $a_n$, satisfies
\[
\left| \sum_{n=1}^m a_n \right| \leq K, \ \forall m.
\]

Suppose another monotone sequence of positive $b_n \to 0$.
Prove that $\sum_{n=1}^\infty a_n b_n$ converges. (Hint: summation by parts)

(b) Hence prove that for $|z| = 1$, except for $z = 1$,
\[
\sum_{n=1}^\infty \frac{z^n}{n} = \log \left\{ \frac{1}{1 - z} \right\}
\]
where the principal branch of logarithm is used.
5. Suppose $f$ is a measurable real-valued function of two real variables $x > 0$ and $y > 0$. Suppose

(i) for every $y > 0, x \mapsto f(x, y)$ on $(0, \infty)$ decreases monotonically to zero as $x \to \infty$.

(ii) for every $x > 0, F(x) = \int_0^\infty f(x, y) \, dy < \infty$.

(iii) $f_1(x, y) = \frac{\partial f}{\partial x}(x, y)$ is a continuous function of $(x, y)$.

Prove

$$F(x) = -\int_x^\infty \int_0^\infty f_1(t, y) \, dy \, dt,$$

for every $x > 0$ and,

$$F'(x) = \int_0^\infty f_1(x, y) \, dy$$

for almost every $x > 0$.

6. For the function $T(x) = \tan(x)$ define the nth iterate $T^n = T \circ T \circ \ldots \circ T$.

(a) Prove that for $y > 0$

$$\lim_{n \to \infty} T^n(1y) = 0$$

(b) Show $\{T^n\}$ is not a normal family in any neighborhood of $x = 0$. 
\[ f \in L^1(0,\infty), \quad f \geq 0 \]

\[
\lim_{n \to \infty} \int_0^\infty \frac{1}{n} \cdot x \cdot f(x) \, dx = \lim_{n \to \infty} \int_0^\infty 1_{[0,n]}(x) \cdot f(x) \, dx
\]

\[ 0 \leq 1_{[0,n]}(x) \cdot x \leq 1 \quad \forall n \text{ and } \forall x \text{ so } \]

\[ g_n = 1_{[0,n]}(x) \cdot f(x) \quad \forall n, x \]

Since \( f \in L^1(0,\infty) \), we can apply Lebesgue Dominated Convergence Theorem which says that

\[
\lim_{n \to \infty} \int_0^\infty g_n(x) \, dx = \int_0^\infty \lim_{n \to \infty} g_n(x) \, dx.
\]

Since \( f(x) \) is finite a.e. \( \lim_{n \to \infty} 1_{[0,n]}(x) \cdot f(x) = 0 \) a.e.

\[ \therefore \int_0^\infty \lim_{n \to \infty} g_n(x) \, dx = 0 = \lim_{n \to \infty} \int_0^\infty (x) \cdot f(x) \, dx \]
Since $u$ is only guaranteed to have a harmonic conjugate in simply connected domains we will consider the following domains separately:

\[ \begin{align*}
\text{on } A & \exists v_1 \text{ s.t. } h = u + iv_1 \text{ is analytic on } A \quad \text{and on } B \exists v_2 \text{ s.t. } u + iv_2 \text{ is analytic on } B. \quad \text{They agree in value on a subset of } A \\
\text{so } h - g \text{ is constant, } h - g = b \quad \text{or } (u + iv_1 - (u + iv_2)) = b
\end{align*} \]

\[ i(v_1 - v_2) = b \]

to claim this analytically continues onto the segment on the negative real axis we need the limits from above and below to be the same, but they differ by this imaginary constant $b$ just as $\log(z^3)$ is analytic on this annulus except that segment where it has an imaginary jump discontinuity of magnitude $2\pi i$.

\[ f = h - \frac{b}{2\pi i} \log(z) \]

will be analytic everywhere on the annulus except the negative real axis but now the discontinuity of the two analytic functions cancel and limits along the negative real axis exist.

\[ \Rightarrow \text{this function has analytic continuation to the whole annulus } A \]

and

\[ \Re f = \Re h - \frac{b}{2\pi} \log |z|^3 \]

or

\[ \Re f = \Re h - \frac{b}{2\pi} \log |z| \]

or

\[ M = \Re f + \frac{b}{2\pi} \log |z| \]
a) \[ \lim_{n \to \infty} \int_{|x| > \frac{1}{n}} \eta \chi(u_n x) \, dx = \lim_{n \to \infty} \int_{|u| > \frac{1}{n}} \eta \chi(u) \, du \]

\[ = \lim_{n \to \infty} \left( \int_{|u| > \frac{1}{n}} \eta \chi(u) \, du - \int_{|u| < \frac{1}{n}} \eta \chi(u) \, du \right) = \lim_{n \to \infty} \left( \int_{|u| > \frac{1}{n}} \eta \chi(u) \, du - \chi_{E_{\text{Lip}}(u_n \eta)} \chi(u) \right) \]

\[ \leq \int_{|u| > \frac{1}{n}} |\eta \chi(u) - \chi_{E_{\text{Lip}}(u_n \eta)} \chi(u)| \, du \leq \chi_{E_{\text{Lip}}(u_n \eta)} \chi(u) \leq \chi_{E_{\text{Lip}}(u_n \eta)} \chi(u) \]

\[ = \lim_{n \to \infty} \int_{|u| > \frac{1}{n}} \eta \chi(u) \, du - \chi_{E_{\text{Lip}}(u_n \eta)} \chi(u) \to 0 \text{ as } n \to \infty \]

By the Lebesgue Dominated Convergence Theorem.

b) \[ \lim_{n \to \infty} \int_{R} \eta \chi(u_n x) g(x) \, dx \overset{m}{=} \lim_{n \to \infty} \int_{R} \eta \chi(u_n) g \left( \frac{u}{n} \right) \, du \]

Note: \[ \int_{R} \eta \chi(u) M = M \int_{R} \eta \chi(u) = M \] so \( \eta \chi(u) M \) is integrable.

Let \( M = \) the bound on \( g \) over \( R \)

Then \( |\eta \chi(u_n) g \left( \frac{u}{n} \right)| \leq \chi_{E_{\text{Lip}}(1)} \cdot M \). So as above apply LDC to get

\[ \lim_{n \to \infty} \int_{R} \eta \chi(u_n) g \left( \frac{u}{n} \right) \, du = \int_{R} \lim_{n \to \infty} \eta \chi(u_n) g \left( \frac{u}{n} \right) \, du \quad \text{since } g \text{ is continuous} \]

\[ \lim_{n \to \infty} g \left( \frac{u}{n} \right) = g \left( \lim_{n \to \infty} \frac{u}{n} \right) = g(u) \]

So \[ \lim_{n \to \infty} \int_{R} \eta \chi(u_n) g \left( \frac{u}{n} \right) \, du = \int_{R} g(u) \chi(u) \, du = g(0) \int_{R} \chi(u) \, du = g(0) \]

\[ \square \]
(1) \( S_{n,m} - S_n \)

\[ \left| \sum_{j=0}^{n} a_j b_j - \sum_{j=0}^{m} a_j b_j \right| = \left| \sum_{j=n+1}^{m} a_j b_j \right| \]

\[ \leq \left| b_{n+1} \right| \sum_{j=n+1}^{m} a_j + \left| \sum_{j=n+1}^{m} ((b_{j+1} - b_j) \sum_{k=j+1}^{m} a_k) \right| \]

\[ \leq 2 b_{n+1} K + \sum_{j=n+1}^{m} (b_j - b_{j+1}) \sum_{k=j+1}^{m} a_k \]

\[ \leq 2 b_{n+1} K + 2 \sum_{j=n+1}^{m} (b_j - b_{j+1}) \Delta \]

\[ = 2K \left[ b_{n+1} + (b_{n+1} - b_{n+1}) \right] \]

so the sequence is cauchy.

(b) \( \forall z \neq 1 \) s.t. \( z \neq 1 \) except \( z = 1 \)

\[ \frac{1}{1-z} = 1 + z + z^2 + \ldots \]

\[ \Rightarrow \int \frac{1}{1-z} = \int 1 + z + z^2 + \ldots \]

\[ \Rightarrow -\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots \]

\[ \Rightarrow \log \left( \frac{1}{1-z} \right) = \sum_{n=0}^{\infty} \frac{z^n}{n} \]