1. Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for a correct application even though you do not give a correct proof.

2. Use a different sheet (or different set of sheets) for each question. Write the problem number and your code number (not your name) on the top of every sheet.

3. Keep scratch work on separate sheets.

4. Unless otherwise stated, you may appeal to a “well-known theorem” in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.
1. Suppose that \( f : [0, \infty) \mapsto [0, \infty) \) is measurable and that \( \int_0^1 f(x) \, dx < \infty \).
Prove that
\[
\lim_{n \to \infty} \int_0^\infty \frac{x^n f(x)}{1 + x^n} \, dx = \int_1^\infty f(x) \, dx .
\]

2. Compute the partial fractions decomposition of \( f(x) = \frac{x^7}{x^3 + 1} \).

3. Let \( f \in L^p[0, 1], g \in L^q[0, 1], h \in L^r[0, 1] \), where \( 1 \leq p, q, r \leq \infty \), \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \). Prove that \( fgh \in L^1[0, 1] \) and \( \|fg\|_1 \leq \|f\|_p \|g\|_q \|h\|_r \).

4. Consider the series:
\[
g(x) = \sum_{n=1}^\infty \frac{x^n}{n!}
\]
(a) Find the domain \( D \) where the series is convergent.
(b) Prove that for any \( k, n \in \mathbb{N} \), \( g(e^{2\pi k}x) = g(x) + p(x) \), for some polynomial \( p \).
(c) Prove that if \( g \) has analytic extension from \( D \) then it has analytic extension to \( \partial D \).
(d) Prove that \( g \) has no analytic extension from \( D \).

5. Suppose \( E_n \) are measurable sets, and there is an integrable function \( f \in L^1(\mathbb{R}) \) such that \( \lim_{n \to \infty} \|\chi_{E_n} - f\|_1 = 0 \). Prove that there is a measurable set \( E \) such that \( f = \chi_E \) a.e.

6. Let \( f \) be a conformal mapping of the domain
\[
\Omega = \{ z : \Re(z) > 0 \} - (0, 1)
\]
onto the domain \( \{ z : \Re(z) > 0 \} \) so that \( f(2) = 2, f'(2) > 0 \). Prove that we have real \( f(3) > 3 \).
\[ \lim_{n \to \infty} \int_0^1 \frac{x^n}{1+x^n} f(x) \, dx = \lim_{n \to \infty} \int_0^1 \frac{x^n}{1+x^n} f(x) \, dx + \lim_{n \to \infty} \int_1^\infty \frac{x^n}{1+x^n} f(x) \, dx \]

hold \quad \left| \frac{x^n}{1+x^n} f(x) \right| \leq \left| f(x) \right| \quad \text{and} \quad \int_0^1 \left| f(x) \right| \, dx < \infty \quad \text{L.D.C.T.}

\[ \lim_{n \to \infty} \int_0^1 \frac{x^n}{1+x^n} f(x) \, dx = \lim_{n \to \infty} \frac{x^n}{1+x^n} f(x) = \int_0^1 f(x) \, dx = 0 \]

Note that for \( x > 1 \)

\[ \frac{x^{n+1}}{1+x^{n+1}} > \frac{x^n}{1+x^n} \quad \text{since} \quad \frac{x+x^{n+1}}{1+x^{n+1}} > 1 \quad \forall \ x > 1. \]

So \( \frac{x^n}{1+x^n} f(x) \) is monotone increasing. Lébesgue Measure Convergence Theorem tells us,

\[ \lim_{n \to \infty} \int_0^1 \frac{x^n}{1+x^n} f(x) \, dx = \int_0^1 \lim_{n \to \infty} \frac{x^n}{1+x^n} f(x) \, dx = \int_0^1 f(x) \, dx = \int_0^1 f(x) \, dx \]

So

\[ \lim_{n \to \infty} \int_1^\infty \frac{x^n}{1+x^n} f(x) \, dx = \int_1^\infty f(x) \, dx \]
\[ f(x) = x^6 + 1 \]
\[ f'(x) = 6x^5 \]

\[ \frac{f'(x)}{f(x)} = \frac{6}{x^6} + \frac{g'(x)}{g(x)} \]
where \( x = a \) is a root of order \( m \) and \( g(x) \)

\[ g(x) (x-a)^m = f(x) \]

So, \( \frac{8x^7}{x^8 + 1} = \frac{1}{2} \cdot \frac{1}{2-x} \)

where \( 2 \) is the 6th root of \( x^6 + 1 \)

\[ \frac{2^7}{2^8 + 1} = \frac{1}{2} \cdot \frac{2}{2 - 2} \cdot \frac{1}{2 - 2} \]

The only step left is to bring back together the roots that go together in the factorization of \( x^6 + 1 = (x+1)(x). \)
Holdes gives us if \( f \in L^p \) and \( g \in L^q \) let \( s = \frac{p}{p + q} \)

Thus \( f^s \in L^p[0,17] \) and similarly \( g^s \in L^q[0,17] \)

Since \( \frac{s}{p} + \frac{s}{q} = 1 \), holdes tells us that \( f^s \cdot g^s \in L^1[0,17] \) and

\[
||f^s \cdot g^s||_1 \leq ||f^s||_{L^p} \cdot ||g^s||_{L^q}.
\]

Or

\[
||f^s \cdot g^s||_1 = \left( ||f^s||_{L^p} \right)^{1/s} \cdot \left( ||g^s||_{L^q} \right)^{1/s} = \left( \int f^p \right)^{1/p} \cdot \left( \int g^q \right)^{1/q} = ||f||_p \cdot ||g||_q.
\]

So

\[
||f^s \cdot g^s||_1 \leq ||f||_p \cdot ||g||_q.
\]

\( \Rightarrow \)

\[
||f \cdot g||_p \leq ||f||_p \cdot ||g||_q \leq ||f||_{L^p} \cdot ||g||_{L^q}.
\]

Let since \( \frac{s}{p} + \frac{s}{q} = \frac{p + q}{p + q} \), \( 1 = \frac{1}{s} + \frac{1}{s} = \frac{1}{p} + \frac{1}{q} = 1 \)

Holdes gives us \( fgh \in L^1[0,17] \) and

\[
||fgh||_1 \leq ||fgh||_{L^1} = ||f||_p \cdot ||g||_q \cdot ||h||_r.
\]
a) \[ \frac{1}{r} = \lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \left( n! \right)^{1/n} \]
\[ = \lim_{n \to \infty} \left( n^n e^{-n} \right)^{1/n} \]
by Stirling's Formula
\[ = \lim_{n \to \infty} n^{n/n^2} \cdot \lim_{n \to \infty} \frac{n}{2^n} = 1 \cdot 1 = 1 \]
So \( \frac{1}{r} = 1 \) or \( r = 1 \), \( D = \text{open and connected} \).

b) \[ g(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \]
\[ g(e^{2\pi ik/2n}) = \sum_{m=0}^{n} e^{2\pi i k 2^{m-n}} \]
\[ g(e^{2\pi i k/2n}) = \frac{e^{2\pi i k} 2^m}{m!} \text{ if } m \geq n, \quad 2^m = 1 \]

So \[ \frac{\sum_{m=n}^{\infty} \frac{z^m}{m!}}{n!} + p(z) \]
\[ = \frac{\sum_{m=0}^{n-1} \frac{z^m}{m!}}{n!} + p(z) + \sum_{m=1}^{\infty} \frac{z^m}{m!} - \frac{2^m}{m!} \]

\[ = g(z) + p(z), \quad p(z) = p'(z) = \frac{\sum_{m=1}^{\infty} \frac{z^m}{m!}}{n!} \]

e) If \( D \) has analytic continuation to an arc on the boundary of the disc,

it will have analytic continuation to every point rational argument away from

\( \partial D \) by the principle of the fundamental theorem. So the extension

can be made to all of \( \partial D \) since these points are

dense in \( \partial D \).

f) By e) \( n \cdot g \) has analytic continuation.

It extends to \( \partial D \) and that continues

d) since this implies \( g \) extends to a region strictly greater than 2 in radius.

from the origin.
\[ \text{Since } f \in L^1(\mathbb{R}) \text{ is measurable,} \]

\[ \text{let } B = \{ x \in \mathbb{R} \mid f(x) \neq 1 \text{ and } f(x) \neq 0 \}, \text{ since } f \text{ is measurable, } B \text{ is a measurable set.} \]

Then \[ \| \chi_{E_n} - f \|_1 = \int_B |\chi_{E_n} - f| \, dx \geq \int_B |\chi_{E_n} - f| \, dx \]

\[ \geq \int_B \min \{ |f(x)|, |1 - f(x)| \} \, dx \]

\[ \text{Since } \lim_{n \to \infty} \| \chi_{E_n} - f \|_1 = 0 \]

\[ \int_B \min \{ |f(x)|, |1 - f(x)| \} \, dx = 0 \quad \text{so } m(\emptyset) = 0 \text{ or} \]

\[ \min \{ |f(x)|, |1 - f(x)| \} = 0 \text{ a.e. on } B \text{ but on } B \text{ fix } f \text{ and } f(x) = 1 \text{ for } x \in B \]

\[ \text{so this is not } Lebesgue \text{ so } m(\emptyset) = 0. \]

Let \[ E = \{ x \in \mathbb{R} \mid f(x) = 1 \} \]

\[ \chi_E = f \text{ a.e. on } \mathbb{R} \text{ and } f(x) = 0 \text{ or } 1 \text{ which is a.e.} \]

\[ E \text{ is measurable since } f \text{ is}. \]
By Riemann Mapping Theorem, there exists an analytic function 

\[ \Psi : \Omega \rightarrow \Omega \]

and \( \Psi : F(\Omega) \rightarrow D \) analytic isomorphism s.t. \( \Psi(0) = \Psi(z) = 0 \)

\[ \Rightarrow \quad \Psi \circ \Psi^{-1}(z) \] is an automorphism of \( \Omega \) and s.t.

\[ \Psi \circ \Psi^{-1}(z) = z \]

\[ \Rightarrow \quad \varphi(z) = \Psi^{-1}(\Psi(z)) \] or that \( f \) is uniquely determined.

Hence \( g(z) = \frac{2}{\sqrt{z^2 + 1}} \) is a conformal mapping s.t.

\[ g(0) = 0 \quad \text{and} \quad g'(z) > 0 \quad \text{so} \quad f = g \]

And \( g(3) = \frac{4}{\sqrt{10}} > 3 \).