1. Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for a correct application even though you do not give a correct proof.

2. Use a different sheet (or different set of sheets) for each question. Write the problem number and your code number (not your name) on the top of every sheet.

3. Keep scratch work on separate sheets.

4. Unless otherwise stated, you may appeal to a "well-known theorem" in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.
1. Suppose that \( f \in L^1(\mathbb{R}) \) is a uniformly continuous function. Show that

\[
\lim_{|x| \to \infty} f(x) = 0
\]

2. Prove there is an entire function \( f \) so that for any branch \( g \) of \( \sqrt{z} \)

\[
\sin^2(g(z)) = f(z)
\]

for all \( z \) in the domain of definition of \( g \).

3. Suppose \( f \) is absolutely continuous on \( \mathbb{R} \), and \( f \in L^1(\mathbb{R}) \). Show that if in addition

\[
\lim_{t \to 0^+} \int_{\mathbb{R}} \left| \frac{f(x + t) - f(x)}{t} \right| \, dx = 0,
\]

then \( f = 0 \) a.e.

4. Let \( \mathcal{H} \) be the domain \( \{ z : -\pi/2 < \Re(z) < \pi/2, \Im(z) > 0 \} \). Prove that \( g = \sin(z) \) is a \( 1:1 \) conformal mapping of \( \mathcal{H} \) onto a domain \( D \). What is \( D \)?

5. Suppose that \( L^{1/2}(\mathbb{R}) \) is the set of all equivalence classes of measurable functions for which

\[
\int_{\mathbb{R}} |f(x)|^{1/2} \, dx < \infty.
\]

(a) Show that it is a metric linear space with the metric

\[
d(f, g) = \int_{\mathbb{R}} |f(x) - g(x)|^{1/2} \, dx,
\]

where \( f, g \in L^{1/2}(\mathbb{R}) \).

(b) Show that with this metric \( L^{1/2}(\mathbb{R}) \) is complete.

6. Suppose that for a sequence \( a_n \in \mathbb{R} \) and any \( z, \Re(z) > 0 \), the series

\[
h(z) = \sum_{n=1}^{\infty} a_n \sin(nz)
\]

is convergent. Show that \( h \) is analytic on \( \{ \Re(z) > 0 \} \) and has analytic continuation to \( \mathbb{C} \).
Let's assume the limit is not zero and I claim we will get a contradiction. If \( f \) lies in \( L'(\mathbb{R}) \):

1. If \( \lim_{x \to \infty} f(x) \neq 0 \), then we assume \( \lim_{x \to \infty} f(x) > 0 \). (If it happens to be \( \lim_{x \to \infty} f(x) < 0 \), a symmetric argument works.)

2. If \( \lim_{x \to \infty} f(x) = 0 \),
   
   \[
   \exists \epsilon > 0 \text{ s.t. } \forall N \exists x_N > N \text{ s.t. } |f(x_N)| > \epsilon.
   \]

   Then \( f \) is u.c. if \& s.l.

   \[
   \forall x_N, \int_{x_N - \delta}^{x_N + \delta} |f| \geq \epsilon \delta \geq \epsilon \delta = \infty \Rightarrow f \not\in L'(\mathbb{R}).
   \]
\[ \sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2} \]
\[ \cos \theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \]
\[ \sin \theta = \frac{\theta}{2} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \]
\[ 1 - \cos \frac{\theta}{2} = \frac{1}{2} \left[ \frac{\theta^2}{2} - \frac{\theta^4}{4!} + \ldots \right] \]

which is a power series of only even powers, so on a branch of \( \sqrt{z} \rightarrow g(z) \)

\[ \sin^2 \left( g(z) \right) = \frac{1}{2} \left[ \frac{2!}{2} z^2 - \frac{4!}{4!} z^2 + \ldots \right] \]

which is entire.
\[ \int |f'(x)| = \int \liminf_{x \to x_0} \left| \frac{f(x+\epsilon) - f(x)}{\epsilon} \right| \, dx \quad \text{by Fatou's lemma,} \\
\leq \liminf_{\epsilon \to 0} \left( \int \frac{|f(x+\epsilon) - f(x)|}{\epsilon} \, dx \right) = \lim_{x \to x_0} \left( \int \frac{f(x+\epsilon) - f(x)}{\epsilon} \, dx \right) = 0 \\
\Rightarrow |f'(x)| = 0 \quad \text{a.e.} \Rightarrow f \text{ is constant a.e. and one } f \text{ is absolutely continuous, but if } f \in L^1(\mathbb{R}) \text{ the only constant function in } L^1(\mathbb{R}) \text{ is 0.} \\
\text{So } f = 0 \text{ a.e.} \]
To prove that \( f = \sin z \) is conformal on this region it suffices to show that \( f' = \cos z \) is non-zero on this region.

**Lemma:**
\[
\cos (z) = e^{i\pi} = -1
\]
\[
\therefore \quad |e^{i\chi y} - e^{-i\chi y}| = |e^{i\chi y} + e^{-i\chi y}|
\]
\[
\therefore \quad e^{y} = e^{-y} \quad \text{or} \quad y = 0 \quad \Rightarrow \quad z \text{ is real}
\]

but there are no real values in our domain.

To prove \( \sin z \) is 1-1 it is convenient to know that

\[
\sin(z) = \sin(\bar{z}) = 2 \sin \left( \frac{z - \bar{z}}{2} \right) \cos \left( \frac{z + \bar{z}}{2} \right)
\]

So \( \sin(z) = \sin(\bar{z}) \) if

\[
2 \sin \left( \frac{z - \bar{z}}{2} \right) \cos \left( \frac{z + \bar{z}}{2} \right) = 0
\]

but \( \cos \left( \frac{z + \bar{z}}{2} \right) \) is not zero since the sum of two numbers with positive imaginary part has positive imaginary part.

So \( \sin \left( \frac{z - \bar{z}}{2} \right) = 0 \)

\[
\therefore \quad \bar{z} - z = k\pi \quad \text{but the only way to get} \quad \frac{z - \bar{z}}{2} = k\pi \quad \text{in our region is if} \quad k = 0 \quad \Rightarrow \quad z = \bar{z} \quad \text{so} \quad \sin(z) = \sin(\bar{z})
\]

or \( \sin(z) \) is 1-1.

Since \( f \) is conformal and 1-1 we know that \( f \) will take the boundary of our region to the boundary of our image.

The image of the line segment that makes up the part of the boundary that is the real line takes:
\[
[\frac{-\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1]
\]

The right boundary of \( D \) are points of the form \( \frac{\pi}{2} + iy \quad (y > 0) \)

\[
\sin \left( \frac{\pi}{2} + iy \right) = \frac{e^{iy} - e^{-iy}}{2i} = \frac{i e^{-y} + ie^{y}}{2i} = e^{y} + e^{-y} \in \mathbb{R}
\]

So both \([-1, \infty)\).

Similarly, \( \sin \left( -\frac{\pi}{2} + iy \right) = (-\infty, -1] \).

So the image of the boundary is \( \mathbb{R} \) so the region must be the upper half plane or lower half plane.

Check rule that \( \sin(i) = \frac{e^i - e^{-i}}{2i} \in \text{upper half plane} \) so the image is the upper half plane.
a) you get symmety (d(x,y) = d(y,x)) and positive i.e. (d(x,y) > 0) is obtained immediately from the corresponding property of absolute value.

Now if d_P(g,h) = 0 \[ \int |g - h|^p = 0 \Rightarrow |f - g|^{1/p} = 0 \text{ a.e.} \]

\[ \Rightarrow P = g \text{ a.e.} \Rightarrow f = g \text{ in equivalence class.} \]

The last property (triangle inequality i.e. look at)

\[ d(f, g) + d(g, h) = \int |f - g|^p + \int |g - h|^p \]

(\star) \[ \int |f - g|^p + |g - h|^p \geq \int |f - g + g - h|^p \]

\[ = \int |f - h|^p \]

\[ \text{[the \ \star \ \text{comes from the fact that } x^{1/p} \text{ is a concave function so]} \]

\[ x^{1/p} + y^{1/p} \geq (x + y)^{1/p} \]

So d_P is a metric.

For linearity we need that linear combinations are contained,

\[ \int (af + bg)^p \leq \int (|af|^p + |bg|^p) \leq \int |a|^p f^p + \beta \int |b|^p g^p \]

which is finite since both terms are finite.

\[ \text{\textbf{Part b:}} \]

\[ \text{\textbf{Not Worth It.}} \]

\[ \text{\textbf{Proof is 3 pages long!}} \]
If \( h(z) = \sum_{n=0}^{\infty} a_n \sin(nz) \) is uniformly convergent on compact subsets of \( E \), then \( h \) is analytic.

Let \( K \) be a compact subset. \( K \subset B(0,r) \) for some \( r \).

For \( x \in K \),
\[
|a_n \sin(nz)| = \left| a_n \frac{e^{in(x+iy)} - e^{-in(x+iy)}}{2i} \right|
\]
\[
\leq |a_n e^{ny}| + |a_n e^{-ny}| \leq |a_n| + |a_n e^{ny}| \quad \text{on our region}
\]

Since \( \sum_{n=0}^{\infty} |a_n \sin nx| \) converges absolutely,

choose \( z = 2i \)

to say \( \sum_{n=0}^{\infty} |a_n \frac{e^{2ny} - e^{-2ny}}{2i}| \geq \sum_{n=0}^{\infty} |a_n| \)

choose \( z = 2i \)

to say \( \sum_{n=0}^{\infty} |a_n \frac{e^{2ny} - e^{-2ny}}{2i}| \geq \sum_{n=0}^{\infty} |a_n e^{ny}| \)

So both terms in (A) converge by Weierstrass M-test.

We have uniform convergence on \( K \).

Now we can apply Shauder reflection to get an analytic continuation of the function to \( C \).