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Introduction

1.1 Purpose of the Course

Most of your university courses focus on new material for you to learn, and test you on how well you have learned and understood it. Your Mathematics courses have emphasized learning algorithms to construct derivatives and integrals, to multiply and invert matrices, and to solve linear equations.

Your objective in this course is different: it is to solve theoretical problems in mathematics and to use rigorous mathematical language to prove your solutions are correct.

For this, you must

- Fully understand the problem, and the mathematical terms used. In particular, make sure you understand
- what you are given (the hypotheses);
- what you are asked to prove (the conclusion);
- the terminology and the notation;
- Figure out why the statement to prove is true.
- Express your idea in a formal mathematical proof.

You will learn techniques you can use to solve problems, but there are no algorithms for this - each problem requires its own idea for a solution. In writing a proof you should:

1. First state what is assumed to be true (the hypotheses).
2. Then state what is to be proved (the conclusion).
3. Then, beginning with the word "proof" provide a sequence of statements, each following logically from the preceding ones and the hypotheses.

4. The final statement should be the conclusion, which your proof has now established as true.

Most of what you will be proving are mathematical statements:

**Definition 1** A *mathematical statement* either defines notation or terminology, or else it is a clear, unambiguous assertion, usually using mathematical terms.

Here are some easy examples of mathematical proofs:

**Example 1**

1. Prove that if a student is in Math 310 (the hypothesis), s/he is registered at the University of Maryland (the conclusion).

   **Proof:** To be in Math 310 you must be registered at the University. Therefore, since the student is in Math 310, s/he is registered at the University. *q.e.d.*

2. Prove that if \( x > 1 \) then \( x^2 > x \).

   **Proof:** \( x^2 - x = x(x - 1) \). Since \( x > 1 \), \( x \) and \( x - 1 \) are both positive. Therefore \( x(x - 1) \) is positive. Therefore \( x^2 > x \). *q.e.d.*

3. If \( n \) is a natural number and \( n > 1 \), prove that \( n^3 - 1 \) is divisible by \( n - 1 \).

   **Proof:** let \( n \) be a natural number and suppose \( n > 1 \). Direct multiplication gives \( (n - 1)(n^2 + n + 1) = n^3 - 1 \). Since \( n^2 + n + 1 \) is a natural number, \( n^3 - 1 \) is divisible by \( n - 1 \). *q.e.d.*

4. Prove that if every student in Math 310 gets at least 7/10 on a homework then the class average is at least 7/10.

   **Proof:** Let the number of students in the class be \( n \) (Establishes notation.). Suppose every student has a grade at least 7/10. Let \( x_i \) be the grade of the \( i \)th student. Then, by definition, the class average is

   \[
   \frac{\sum_{i=1}^{n} x_i}{n}.
   \]

3
Since each $x_i \geq 7$, therefore
\[
\frac{\sum_{i=1}^{n} x_i}{n} - 7 = \frac{\sum_{i=1}^{n} x_i}{n} - \frac{7n}{n} = \frac{\sum_{i=1}^{n} (x_i - 7)}{n} \geq 0.
\]
Therefore the class average is at least $7/10$.

5. **The Division Proposition**: If $p < n$ are natural numbers then for some natural number $m < n$ and for some integer $r$ with $0 \leq r < p$, 
\[
n = mp + r.
\]
We call $r$ the **remainder**.

**Proof**: Let $p < n$ be natural numbers. If $k > n$ then for any $r \geq 0$
\[
kp \geq k > n.
\]
Thus if $mp + r = n$ it must be true that $mp \leq n$ and so $m \leq n$.
Since $1 \leq n$ we can choose $m$ to be the largest natural number such that $mp \leq n$.
Then
\[
n - mp \geq 0.
\]
On the other hand, if $n - mp \geq p$ then $(n - mp) - p > 0$ and so
\[
n - (m + 1)p = (n - mp) - p > 0.
\]
Thus $m$ was not the largest natural number which multiplied by $p$ gave an answer at most $n$, and we assumed it was. Thus $n - mp < p$ and we may set $r = n - mp$. q.e.d.

Everyone, even at an early age, learns what it means for a statement to follow logically from others, and sometimes it will be obvious that the statement is correct. So, writing proofs and solving problems does not require you to learn to think logically, since you already know how to do that.

However, this course is about **Mathematics**. Most of the problems are about **mathematical objects**, and so you must get to know and understand these objects well and you must acquire the ability to **read and write in mathematical language**, since that is the language in which the problems are posed and in which your proofs must be expressed.

**Tips for Using Mathematical Language**:
1. Mathematical assertions are **always** either true or false. There is no 'middle ground'.
2. Mathematical language is about **mathematical objects** such as sets, numbers, functions, ···.


3. When writing mathematics it is almost always useful to label the objects under consideration. This is called "establishing notation".

4. In Mathematics words and symbols must be carefully and precisely defined, with no ambiguity.

5. **When you write a mathematical statement it must say exactly what you mean:**

   (a) Even if you feel that the reader will understand what you wrote although you have not said it exactly, **in this course that is not acceptable.**

   (b) Consequently you will need to always write in clear, unambiguous complete sentences, **which say precisely what you mean.**

**Homework requirements**

1. Your answers must be a sequence of mathematical statements as described above, and you will get a **low grade** if they do not meet those requirements, even if they give the impression that you understand the material.

2. In particular your answers **must** be:

   (a) typed or written legibly in ink;

   (b) clear, precise, and unambiguous;

   (c) without excess verbiage;

   (d) written in complete, grammatical sentences.

3. Each statement must follow logically from the previous ones, with an explanation as to why.

4. Your homework answers may rely on facts as outlined below, but you must in each case indicate which fact you are using and give the explicit reference:

   (a) any statement already proved in class;

   (b) any statement proved in an earlier part of the text;

   (c) any statement in any earlier exercise.

**Tips for Writing Proofs** from Elizabeth (a former Math 310 student):

1. Learning to write proofs is not like learning how to multiply matrices, and it can take mental effort and time.

2. Remain calm; if you put enough time into this class, you will figure out how to write proofs.

3. Give yourself plenty of time to work on a proof.
4. Before you start the proof, look back at the given definitions and lemmas that you think might be useful for this proof. Make sure you fully understand the definitions and lemmas and see if you can find a way to connect them back to what you are trying to prove.

5. Restate definitions in your proofs, so you are less likely to use a definition incorrectly.

6. Working in groups is fine, but make sure you try each exercise thoroughly on your own first. You will not have your group to help when it comes time to take the test.

7. Never hesitate to ask Professor Halperin a question when you are stuck: either in class, in his office, or by email. He will welcome your interest: I know!

8. Most importantly: This class is designed to help you learn how to write proofs. While it is important you understand the concepts, you must be able to use what you learn to complete a correctly written proof in order to succeed.

**In Summary, to succeed** in this course you will need to:

1. **Absolutely understand the definitions** of the words and symbols you run into.

2. Be **totally comfortable** with the meaning and properties of the objects the words refer to.

3. **Understand** and be able to explain to others the precise meaning of each mathematical statement you encounter.

4. Write so that each sentence has a **single** meaning which will be clear to any reader.

5. Hand in each homework and **master** the assigned readings in the text.

6. Get help from me in class, by email, or in office hours, when you do not understand something.

7. Join a study group, but never as a freeloader. If you freeload your home work score may be OK, but your tests and exams will NOT be.

**Students who invest this time and effort usually do well. Students who do not make this effort usually do not.**
Chapter 2

Mathematical Proof

2.1 The Language of Mathematics

In Mathematics, a definition assigns a precise meaning to a word or symbol.

Definitions and notation are essential tools in keeping our language unambiguous and terse and, as with any new language, it is essential that you internalize the definitions and notation rather than simply memorize them. Indeed, when learning to speak or write a new language you need to be able to use the words spontaneously without having to call up each corresponding English word and then translate it. In the same way, you need to be able to speak/write mathematics, not just remember the dictionary of definitions.

You learn a second language most easily by speaking it with others to whom it comes naturally. You learn to drive a car by driving it and to walk by walking. You learn to write/speak mathematics by writing it and presenting it and getting feedback when you get it right and how to correct it when you don’t. The golden rule when writing: never write anything whose meaning is unclear to yourself! You can also use this text to find many detailed examples of how to write a proof correctly.

The language of mathematics consists of assertions about mathematical objects. Mathematical objects include the natural numbers, the integers, the rationals, the real numbers, sets, maps, functions and many other things. Mathematical language has its own vocabulary: for example, two words which appear throughout mathematics are the words set and sequence:

Definition 2

1. A set is a specified collection of distinct objects, abstract or concrete, called its elements. An element $x$ in a set $S$ is said to belong to $S$, and
we denote this by \( x \in S \). We often write

\[
S = \{x, y, z, \ldots\},
\]

where \( x, y, z \ldots \) are the elements of \( S \).

2. If \( x \) is an element in a set \( S \) this is denoted \( x \in S \).

3. A **sequence** starting at \( k \) is a **list** \( (x_n)_{n \geq k} \) of objects, possibly with repetitions, and indexed by the integers \( n, n \geq k \). If \( (x_n)_{n \geq k} \) is a sequence, then \( x_n \) is the \( n \)th **term** in the sequence.

**Note:** The objects listed in a sequence form a set, as seen in the first two of the following examples.

**Example 2**

1. The list \( 0, 1, 0, 1, 0, 1, \ldots \) is a sequence. The corresponding set is the collection \( \{0, 1 \} \) which only has two elements.

2. The list \( 2, 4, 2, 4, 6, 2, 4, 6, 8, 2, 4, 6, 8, 10, \ldots \) is a sequence. The corresponding set is the collection of even natural numbers.

3. A sequence \( (x_n)_{n \geq 1} \) of numbers is defined by

\[
x_n = 2^n + n^2.
\]

4. A sequence of fractions \( (y_n)_{n \geq 2} \) is defined by

\[
y_n = \frac{n}{n - 1}.
\]

Mathematical language also relies heavily on symbols to express assertions and proofs in a short and clear manner, but in any specific proof each symbol must have a precise meaning and the same symbol may never be used with two different meanings. In particular, we fix the following symbols for the entire course.

**Example 3  Basic notation**

1. \( N \) will denote the set of natural numbers, and we write \( N = \{1, 2, 3, \ldots\} \).

2. \( Z \) will denote the set of integers: \( Z = \{0, \pm 1, \pm 2, \pm 3 \ldots\} \).

3. \( Q \) will denote the set of rational numbers: these are the real numbers which can be written \( p/q \) with \( p \in Z \) and \( q \in N \). Note that \( p/q = r/s \) if and only if \( ps = rq \).

4. \( R \) will denote the set of real numbers.
5. \( \mathbb{C} \) will denote the set of complex numbers.

6. The **absolute value** of a real number \( x \) is denoted by \( |x| \) and is defined by:

\[
|x| = \begin{cases} 
    x, & x \geq 0, \\
    -x, & x < 0
\end{cases}
\]

Note that \( |x| \geq 0 \) for all \( x \in \mathbb{R} \).

7. **q.e.d.** stands for quid erat demonstrandum, which is Latin for "what was to be proved". It is used at the end of a proof to signal that the proof is complete.

**Definition 3**

1. An integer \( n \) is **even** if it is divisible by 2.

2. An integer \( n \) is **odd** if it is not even.

3. A **prime number** is a natural number \( n \) which is larger than 1 and is not divisible by any natural number except 1 and \( n \).

4. A **positive** real number means a number \( x \) such that \( x > 0 \). Thus zero is **not a positive number**!

5. A **negative** real number means a number \( x \) such that \( x < 0 \). Thus zero is **not a negative number**!

6. A number \( x \) is **bigger** than a number \( y \) if \( x - y \) is positive. (**CAUTION:** Note that \(-1\) is bigger than \(-2\)!)  

7. If \( x \) is a real number and \( n \) is a natural number then \( x^n \) is the real number obtained by multiplying \( x \) with itself \( n \) times.

8. **Let. . . .:** This is a statement which **defines** some terminology or **establishes notation**. As an example: Let "\( n \)" be a natural number greater than 5.

9. **Theorem, proposition, lemma:** These are names for mathematical statements that are going to be proved.

**Example 4** "Sigma" and "Pi" notation

1. The "**Sigma**" notation is used for sums: If \( v_i \) are numbers or vectors (or anything else we know how to add) then

\[
\sum_{i=1}^{n} v_i = v_1 + v_2 + \cdots v_n.
\]

More generally,

\[
\sum_{i=1}^{n} v_{ki} = v_{k1} + v_{k2} + v_{k3} + \cdots v_{kn}.
\]
In this notation the "$i$" just indexes the terms being added or multiplied, and we could use any letter instead without changing the meaning. Thus

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} v_i = \sum_{q=1}^{n} v_q = v_1 + v_2 + \cdots + v_n.$$  

In other words, the "$i$" is a dummy variable just as in calculus, where we have

$$\int f(x)dx = \int f(u)du$$

2. The "Pi" notation is used for products:
   If $v_i$ are numbers then
   $$\prod_{i=1}^{n} v_i = v_1 \cdot v_2 \cdot \cdots \cdot v_n.$$  

More generally,
   $$\prod_{i=1}^{n} v_{k_i} = v_{k_1} \cdot v_{k_2} \cdot v_{k_3} \cdots v_{k_n}.$$  

3. In particular, if $C$ is a constant, then
   $$\sum_{i=1}^{n} C = C + \cdots + C \text{ (n times)} = nC$$
   and
   $$\prod_{i=1}^{n} C = C \cdots C = C^n.$$  

Theorem 1 (Difference Theorem) For any real numbers $a, b \in \mathbb{R}$ and any $n \in \mathbb{N}$,

$$b^n - a^n = (b - a) \sum_{i=0}^{n-1} b^i a^{n-1-i}.$$  

Proof: Note that

$$b(\sum_{i=0}^{n-1} b^i a^{n-1-i}) = \sum_{i=0}^{n-1} b^{i+1} a^{n-1-i} = \sum_{j=0}^{n-1} b^{j+1} a^{n-1-j}.$$  

In the second expression set $j + 1 = i$. Then

$$\sum_{j=0}^{n-1} b^{j+1} a^{n-1-j} = \sum_{i=1}^{n} b^i a^{n-i},$$  

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and so
\[ b \left( \sum_{i=0}^{n-1} b^i a^{n-1-i} \right) = \sum_{i=1}^{n} b^i a^{n-i} = b^n + \sum_{i=1}^{n-1} b^i a^{n-i}. \]

On the other hand,
\[ (-a) \left( \sum_{i=0}^{n-1} b^i a^{n-1-i} \right) = -\sum_{i=0}^{n-1} b^i a^{n-i} = -a^n - \sum_{i=1}^{n-1} b^i a^{n-i}. \]

Adding these two lines gives the equation of the Theorem.
q.e.d.

2.2 Logical statements

In Mathematics, you will frequently encounter statements about statements! Moreover, just as we use symbols to represent unspecified numbers, functions, and matrices, so also we may use symbols to represent statements!

**Example 5**
Here are four statements:

1. Statement A: John and Mary are students in Math 310.
2. Statement B: John and Mary are registered as students in UMD.
3. A implies B.
4. B implies A.

In this example, The statement A implies B is true, and the statement B implies A is false.

Here are some frequently used logical statements and expressions:

1. "If A then B." This assertion states that if A is true then B is true. Here A is the hypothesis or assumption (both words mean the same thing), and B is the conclusion. This is often denoted by

\[ A \Rightarrow B. \]

It may also be phrased as
(a) A implies B, or as
(b) B if A.

2. A only if B. This means that A is not true unless B is true. In other words, if B is not true then A is not true.
3. "A if and only if B" This means that if B is true then so is A, and that if B is not true then A is not true. In other words, A is true if and only B is true. This is often denoted by

\[ A \Leftrightarrow B. \]

4. "For each \( x \in S \) there exists a \( y \) such that A is true" means that for each choice of \( x \) there exists a \( y \) (\( y \) will usually depend on \( x \)) with the properties prescribed by A.

5. The "converse" to the statement "If A is true then B is true" is the statement "if B is true then A is true".

6. The "contrapositive" to the statement "A is true if B is true" is the statement "B is not true if A is not true".

The next Lemma shows that a statement and its contrapositive are equivalent! The proof relies on the fact that in mathematics, every statement is either true or false.

**Lemma 1**

1. The statement "A implies B" is true if and only if its contrapositive is true.

2. \( A \Rightarrow B \) and \( B \Rightarrow A \) \( \Leftrightarrow \) "A is true if and only if B is true".

**Proof:**

1. Suppose that A implies B. Then if B is not true, A cannot be true! Thus if the statement A implies B is true, so is its contrapositive. On the other hand, suppose the contrapositive is true. Then if A is true it cannot be that B is not true. Thus B is true and so A implies B.

2. By definition "A implies B" and "B implies A" together are the same as "A is true if and only if B is true".

q.e.d.

### 2.3 Mathematical statements

Mathematical statements are statements about mathematical objects. They may be definitions, or logical statements, and can express a complicated idea in a few words or symbols, as the following examples show. Thus until one gets used to the language it really can take a mental effort to understand a mathematical statement.
1. **Statement:** Let \((x_n)\) be the sequence of rational numbers defined by \(x_n = 1 - 1/n\). Then for each \(k \in \mathbb{N}\) there is some natural number \(N\) such that \(|1 - x_n| < 1/2k\) if \(n \geq N\).

In this example the natural number \(N\) depends on the choice of \(k\)! In fact,

\[
|1 - x_n| = |1 - (1 - 1/n)| = |1/n| = 1/n.
\]

Thus if \(n \geq N\) then \(|1 - x_n| = 1/n \leq 1/N\). Therefore if \(N = 3k\) then for \(n \geq N\), \(|1 - x_n| \leq 1/3k < 1/2k\).

2. **This example illustrates the importance of using words carefully and correctly.**

Let \((x_n)\) be the sequence of integers defined by \(x_n = (-1)^n\). Consider the following two assertions

(a) **Statement:** There is an \(\varepsilon \in \mathbb{R}\), and there is an \(N \in \mathbb{N}\) such that \(|x_n - 0| < \varepsilon\) for \(n \geq N\).

To see that the first is true pick \(\varepsilon = 3\) and \(N = 1\). Then for \(n \geq N\),

\[
|x_n - 0| = |(-1)^n - 0| = 1 < 3 = \varepsilon.
\]

(b) **Statement:** For each \(\varepsilon > 0\) there is an \(N \in \mathbb{N}\) such that \(|x_n - 0| < \varepsilon\) for \(n \geq N\).

The first is correct and the second is false.

To see that the first is true pick \(\varepsilon = 1/2\) and \(N = 1\). Then for all \(n\),

\[
|x_n - 0| = 1
\]

which is not less than 1/2.

3. **Statement:** For each real number \(a\) and for each \(\varepsilon > 0\) there is a \(\delta > 0\) such that if \(|x - a| < \delta\) then \(|x^2 - a^2| < \varepsilon\).

In this example \(\delta\) will depend on both \(a\) and \(\varepsilon\): a larger \(a\) and a smaller \(\varepsilon\) will require a smaller \(\delta\).

4. **Statement:** If \(x = 2\) and \(y = 4\) then \(x + y = 6\). This assertion is true.

However, the converse assertion: "if \(x + y = 6\) then \(x = 2\) and \(y = 4\)" is false, because if \(x = 1\) and \(y = 5\) then \(x + y = 6\). (This is called a counterexample)
5. **Statement:** Suppose $x$ and $y$ are natural numbers. Then $xy = 2$ only if either $x \geq 2$ or $y \geq 2$.

**Proof:** The hypothesis says that if $xy = 2$ then one of $x$ and $y$ must be at least $2$. But if neither $x$ nor $y$ is at least $2$ then both must be $1$. In this case $xy = 1$, contrary to our hypothesis. Thus one of $x$, $y$ must be at least $2$. q.e.d.

6. By contrast, the statement $xy = 2$ if and only if one of $x \geq 2$ and $y \geq 2$ is false, since it means two things:

   (a) If $xy = 2$, then $x \geq 2$ or $y \geq 2$, AND
   (b) If $x \geq 2$ or $y \geq 2$ then $xy = 2$.

   But if $x = y = 2$ then $xy = 4$ and so the second statement is not true.

7. This example highlights the importance of "getting the order right", with two similar statements:

   (a) For each student in this class there is a date on which that student was born.
   (b) For each date there is a student in the class who was born on that date.

   The first statement is true and the second is false.

8. **Statement:** A natural number $n$ is odd if and only if it has the form $n = 2k - 1$ for some natural number $k$.

   **Proof:** If $n$ has the form $n = 2k - 1$ for some natural number $k$, then it is not divisible by $2$. Therefore $n$ is odd.

   On the other hand, any natural number, when divided by $2$ has a remainder of either $0$ or $1$. If the remainder is $0$ then the number is divisible by $2$, and so it is even.

   Thus if $n$ is odd, when divided by $2$ it has a remainder of $1$. In this case $n = 2m + 1$ where $m = 0$ or $m \in \mathbb{N}$. Set $k = m + 1$. Then $k \in \mathbb{N}$ and $n = 2k - 1$. q.e.d.

9. **Statement:** For every rational number there is a natural number which is larger.

   **Proof:** The rational number must have the form $p/q$ in which $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. (Establishes notation)
If \( p \leq 0 \) then \( p/q < 1 \).
Otherwise \( p \in \mathbb{N} \) and \( p/q \leq p < p + 1 \). \textbf{q.e.d.}

10. \textbf{Statement:} For every positive rational number, \( a \), there is a natural number \( n \) such that \( 1/n < a \).

\textbf{Proof:} Write \( a = p/q \) with \( p, q \in \mathbb{N} \).
Then \( 1/(q+1) < 1/q \leq p/q \). Set \( n = q + 1 \). \textbf{q.e.d.}

\textbf{Exercise 1} 1. For each statement below decide if it is true or false and either provide a proof that it is true or give an example to show that it is false.

(a) If \( x \in \mathbb{Z} \) and \( x \geq 0 \) then \( x \in \mathbb{N} \).
(b) If \( p, q \in \mathbb{Z} \) and \( p + q \in \mathbb{N} \) then \( p \in \mathbb{N} \).

(c) If \( x \in \mathbb{N}, y \in \mathbb{Z} \), and \( z \in \mathbb{C} \), then \( x, y, z \in \mathbb{R} \).

(d) If \( p, q \in \mathbb{N} \), and \( p \geq 2 \) and \( q \geq 6 \), then \( p + q \) is even.

2. Describe what is wrong with each of the following statements:

(a) Laptops are popular among college students. Therefore at least one student at Maryland has a laptop.

(b) It is true that \( u = 6 \) for every \( x, y \in \mathbb{Q} \) such that \( x + y < u \).

3. Does (a) or (b) correctly add "such that" to the statement: "every student has a height \( h \)?

(a) Every student has a height \( h \) such that \( h \leq a \) where \( a \) is the height of the tallest student.

(b) Every student has a height \( h \) such that every student is in a class.

4. Is the following statement true or false? Provide a proof for your answer.
"Suppose \( A \) and \( B \) are natural numbers, and that \( A = 5 \) if and only if \( B = 2 \). If \( B \neq 2 \), then \( A = 7 \).

5. What is the contrapositive of the following statements?

(a) \( A \neq 8 \) if \( B = 9 \).

(b) \( A = 8 \) if \( B \neq 9 \).

6. What is the converse of the following statements?
7. What is the hypothesis and what is the conclusion in the following assertions?

(a) The cube of an odd natural number is odd.

(b) For every $\varepsilon > 0$ there is some $p \in \mathbb{N}$ such that $1/p \leq \varepsilon$.

8. What is the contrapositive of the statement: For every $\varepsilon > 0$ there is a point $(x, y)$ in the plane whose distance from the origin is between $\varepsilon/2$ and $\varepsilon$.

9. What is the converse of the following statement about natural numbers: $n, m$?

If $n > m$ then for some $k \in \mathbb{N}$, $km > n$.

Is the original statement true? Is the converse true? Provide proofs that show your answers are correct.

10. What is the converse of the statement "$A$ implies $B$"? Construct an example where a statement is false but the converse is true.

11. What is the converse of the statement "$A$ is true if and only if $B$ is true."

What is the difference in meaning between the original statement and the converse?

12. Is (a) or (b) a proof that the following statement is wrong? **Statement:** All ducks are red.

(a) There exists a yellow duck, $A$. Therefore not all ducks are red.

(b) We have not seen all ducks. Therefore some may not be red.

13. Prove that if $k \in \mathbb{N}$ and $k^2$ is even then $k^2$ is divisible by 4.

14. Suppose $x, y$ are positive real numbers and $n \in \mathbb{N}$.

(a) Show that $x < y$ if and only if $x^n < y^n$.

(b) If $x < y$ show that if $n > 1$ then $y^n - x^n < n(y - x)y^{n-1}$.

15. In each of the following statements, replace the blanks by words/expressions from the list:

let, therefore, if, only if, only, since, therefore, some, then, by, even, implies, because, each, not, true, false, hypothesis, zero, one, theorem, follows so that the statements are true. Then supply a proof.
(a) ____ n is a positive natural number ____ n + 1 is divisible ____ a prime number.

(b) ____ ____ natural number divides a fixed integer then that integer is ____.

(c) ____ S be a set with a single element. ____ there is ____ ____ sequence starting at ____ listing elements of S.

(d) The statements "A implies B" and "B is ____ true ____ A is ____ true" are equivalent.

(e) ____ x < y be real numbers. ____ y < 0 ____ |y| < |x|.

16. In each of the following, replace the blanks by words/expressions from the list in the previous problem to provide a proof of the given statement.

(a) **Statement**: The sum of an even number of odd integers is even.

**Proof**: ____ k be the number of integers and ____ nᵢ be the iᵗʰ integer.

____ k is even, k = 2m, where m is an integer.

____ nᵢ is odd, nᵢ = 2mᵢ − 1, where mᵢ is an integer.

___ \[ \sum_{i=1}^{k} nᵢ = \sum_{i=1}^{2m} (2mᵢ - 1) = 2 \sum_{i=1}^{2m} (mᵢ) - \sum_{i=1}^{2m} 1 = 2 \sum_{i=1}^{2m} (mᵢ) - 2m. \]

___ \[ \sum_{i=1}^{k} nᵢ \] is ____.

q.e.d.

(b) **Statement**: If k ∈ N and k² is divisible by 3, then k² is divisible by 9.

**Proof**: ____ r be the remainder when k is divided by 3.

____ k = 3p + r for ____ p ∈ Z.

____ k² = 9p² + 6pr + r².

Since by ____ , k² is divisible by 3, it follows that r² is divisible by 3. But r is one of 0, 1, 2 and 1 = 1² and 4 = 2² are not divisible by 3.

____ r = 0 and k = 3p is divisible by 3.

q.e.d.

(c) **Statement**: For any natural numbers p and n, \((p + 1)^n - p^n \geq n\).

**Proof**: By the Difference Theorem,

\[ (p + 1)^n - p^n = (p + 1 - p) \sum_{i=0}^{n-1} (p + 1)^i p^{n-i-1} = \sum_{i=0}^{n-1} (p + 1)^i p^{n-i-1}. \]

____ \( (p + 1)^i \geq 1 \) and ____ \( p^{n-i-1} \geq 1, \)

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\[ \sum_{i=0}^{n-1} (p+1)^i p^{n-i-1} \geq \sum_{i=0}^{n-1} 1 \geq n. \]

\[ q.e.d. \]

2.4 Proof by Contradiction

This is a method to prove that a statement, A, is true by assuming it is false and showing that assumption leads to a contradiction with known facts. In detail, here are the Steps:

1. Assume that A is false.
2. Deduce that statement, B, known to be true must then also be false.
3. Since B is known to be true, conclude that A cannot be false.
4. Therefore A must be true.

Here are some examples of this method:

Example 7 Proof by contradiction

1. Prove that there are infinitely many prime numbers.

   **Proof:** Suppose the statement "there are infinitely many prime numbers" is false. Then there would be only finitely many prime numbers. Thus there would be a largest prime number \( p \).

   Let \( n \) be the number obtained by first multiplying all the natural numbers from 1 to \( p \) and then adding 1. \( \text{(Establishes notation)} \)

   Then \( n > p \).

   Dividing \( n \) by any natural number \( k \leq p \) gives a remainder 1. Therefore \( n \) is not divisible by any natural number \( k \leq p \).

   But \( n \) must be divisible by some prime number \( q \).

   Therefore there must be a prime number \( q > p \).

   This contradicts the hypothesis that \( p \) was the largest prime number.

   Therefore there must be infinitely many primes. \( q.e.d. \)

   **Remark:** This proof was discovered by Euclid about 300 BC.

2. Prove that there is not a smallest positive rational number.
Proof: Suppose the statement "there is not a smallest positive rational number" is false. Then there is a smallest positive rational number, $x$. Since $x$ is a positive rational number, it must be of the form $p/q$, with $p, q \in \mathbb{N}$. But

$$\frac{p}{q+1} - \frac{p}{q} = \frac{pq - p(q + 1)}{q(q + 1)} = -\frac{p}{q(q + 1)} < 0.$$  

Thus

$$\frac{p}{q+1} < \frac{p}{q},$$

and so $p/q$ is not the smallest positive rational number. This is a contradiction, and proves that there is not a smallest positive rational number. q.e.d.

Exercise 2 Proof by contradiction

1. Show that there is not a largest even natural number.

2. Show that there is not a largest rational number of the form $\frac{p}{p+1}$ with $p \in \mathbb{N}$.

3. Show that there is not a largest negative rational number.

4. Prove that 2 is not the square of a rational number.

5. Prove that the equation $x^3 + 4x^2 - 3x + 17 = 0$ can have at most 3 distinct roots.

2.5 Proof and Construction by Induction

Induction is used to prove that every statement $S(n)$ in an infinite sequence of statements is true. It is based on the following induction principle, which we will take as true without proof!

Induction Principle: Suppose given a sequence of statements $S(n)$, one for each $n \in \mathbb{N}$. Suppose that

1. $S(1)$ is true.

2. Whenever $S(n)$ is true then also $S(n + 1)$ is true.

Then $S(n)$ is true for all $n \in \mathbb{N}$.

Example 8 Examples of proof by induction
1. **Statement:** If \( n, m \in \mathbb{N} \) and \( x \) is a non-zero real number then 
\[
x^{n+m} = x^n x^m.
\]

**Proof:** We prove this by induction on \( n \), and set 
\[
S(n) : \text{For every } m, \ x^{n+m} = x^n x^m
\]
Then \( S(1) \) reads for every \( n \in \mathbb{N} \), \( x^{n+1} = x^n x \), which is true by definition. 
Now suppose \( S(n) \) is true for some \( n \). Then because \( S(n) \) is true, 
\[
x^{(n+1)+m} = x^{n+1} x^m = x^n x x^m = x^n x^m = x^{n+1} x^m.
\]
Thus \( S(n+1) \) is true, and so by induction, \( S(n) \) is true for all \( n \). \( \text{q.e.d.} \)

2. **Statement:** If \( n, m \in \mathbb{N} \) and \( x \) is a non-zero real number then

\[
(x^n)^m = x^{nm}.
\]

**Proof:** We prove this by induction on \( m \), and set 
\[
S(m) : (x^n)^m = x^{nm} \quad \text{for every } n \in \mathbb{N},
\]
Then \( S(1) \) reads for any \( n \in \mathbb{N} \), \((x^n)^1 = x^n\), which is true by definition. 
Now suppose \( S(m) \) is true for some \( m \). Then because \( S(m) \) is true, and using the first Statement above, we obtain
\[
(x^n)^{m+1} = (x^n)^m x^n = x^{nm} x^n = x^{nm+n} = x^{n(m+1)}.
\]
Therefore \( S(m+1) \) is true, and so by induction \( S(m) \) is true for all \( m \). \( \text{q.e.d.} \)

3. **Statement:** For every \( n \in \mathbb{N} \),

\[
\sum_{k=1}^{k=n} k^3 = \left( \frac{n(n+1)}{2} \right)^2.
\]

**Proof:** We prove this by induction on \( n \), with \( S(n) \) the equation above.

When \( n = 1 \) both sides of the equation equal 1 and so they equal each other.

Suppose now by induction that \( S(n) \) is true for some \( n \).

Then for this \( n \), since \( S(n) \) is assumed to be true, 
\[
\sum_{k=1}^{k=n+1} k^3 = \sum_{k=1}^{k=n} k^3 + (n+1)^3 = \left( \frac{n(n+1)}{2} \right)^2 + (n+1)^3.
\]
Factoring out \((n+1)^2\) gives

\[
\sum_{k=1}^{n+1} k^3 = (n+1)^2 \left( \frac{n^2}{4} + n + 1 \right) = (n+1)^2 \left( \frac{n^2 + 4n + 4}{4} \right) = \left( \frac{(n+1)(n+2)}{2} \right)^2.
\]

Therefore \(S(n+1)\) is true, and the formula follows by induction. \textit{q.e.d.}

4. \textbf{Statement:} For any \(n \in \mathbb{N}\),

\[
\sum_{k=1}^{n} k \leq \sum_{k=1}^{n} k^2.
\]

\textbf{Proof:} When \(n = 1\) the inequality reduces to \(1 \leq 1\), which is true.

Suppose the inequality holds for some \(n\).

Observe that \((n+1)^2 = (n+1)(n+1) = n^2 + 2n + 1 \geq n + 1\).

Therefore

\[
\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) \leq \sum_{k=1}^{n} k^2 + (n+1)^2 = \sum_{k=1}^{n+1} k^2.
\]

Therefore the inequality holds for \(n+1\), and so by induction the inequality holds for all \(n\). \textit{q.e.d.}

An important application of proof by induction is the \textbf{Binomial Theorem}, which requires a definition and some more notation.

\textbf{Definition 4} 1. For any natural number \(n\), \textbf{\(n\) factorial} is the product of all the natural numbers from 1 to \(n\):

\[
n! = \prod_{k=1}^{n} k.
\]

Additionally we define 0 factorial to be 1: \(0! = 1\).

2. We write

\[
\binom{n}{i} = \frac{n!}{i!(n-i)!}
\]

These numbers are called \textbf{binomial coefficients}.

Now we can state the:

\textbf{Theorem 2 (Binomial theorem)} For any real numbers \(a\) and \(b\), and for any \(n \in \mathbb{N}\),

\[
(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.
\]
To prove this we first need to establish an important property of the binomial coefficients:

**Lemma 2** For any natural number, \( n \), and any natural number \( i \leq n + 1 \),

\[
\binom{n}{i-1} + \binom{n}{i} = \binom{n+1}{i}.
\]

**Proof:** First, we use the definition above to rewrite the left hand side as

\[
\frac{n!}{(i-1)!(n-(i-1))!} + \frac{n!}{i!(n-i)!}.
\]

Multiply the numerator and the denominator of the first term by \( i \) to get

\[
\frac{n!}{(i-1)!(n-(i-1))!} = \frac{n! \cdot i}{i!(n+1-i)!}.
\]

Then multiply the numerator and the denominator of the second term by \( n+1-i \) to get

\[
\frac{n!}{i!(n-i)!} = \frac{n! \cdot (n+1-i)}{i!(n+1-i)!}.
\]

Finally, add together to get

\[
\frac{n!}{(i-1)!(n-(i-1))!} + \frac{n!}{i!(n-i)!} = \frac{n! \cdot (n+1-i)}{i!(n+1-i)!} = \frac{(n+1)!}{(i)!(n+1-i)!}.
\]

q.e.d.

**Proof of the binomial theorem:** We prove this by induction on \( n \).

When \( n = 1 \) the statement reduces to \( a + b = a + b \).

Next, assume the statement is true for \( n \). Then it follows that

\[
(a + b)^{n+1} = (a + b)(a + b)^n = (a + b) \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.
\]

Multiplying separately by \( a \) and \( b \) and adding, we get

\[
(a + b)^{n+1} = \sum_{i=0}^{n} \binom{n}{i} a^{i+1} b^{n-i} + \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i+1}.
\]

We consider the two terms separately. The second term may be written as

\[
\sum_{i=0}^{n} \binom{n}{i} a^{i+1} b^{n-i+1} = b^{n+1} + \sum_{i=1}^{n} \binom{n}{i} a^i b^{n-i+1}.
\]

The first term may be written as

\[
\sum_{i=0}^{n-1} \binom{n}{i} a^{i+1} b^{n-i} + a^{n+1}.
\]
For this first term set \( k = i + 1 \). Then as \( i \) runs from 0 to \( n - 1 \), \( k \) runs from 1 to \( n \), and

\[
   n - i = (n + 1) - (i + 1) = n + 1 - k.
\]

Thus the first term may be rewritten as

\[
   \sum_{k=1}^{n} \binom{n}{k-1} a^k b^{n+1-k} + a^{n+1}.
\]

Now in this sum, \( k \) is a "dummy variable". We could have used any symbol such as \( m, n, p, \ldots \). In particular we could have used \( i \). Thus this term may be rewritten as

\[
   \sum_{i=1}^{n} \binom{n}{i-1} a^i b^{n+1-i} + a^{n+1}.
\]

Finally, adding the two terms together we get

\[
   (a + b)^{n+1} = a^{n+1} + \sum_{i=1}^{n} \left( \binom{n}{i-1} \right) a^i b^{n+1-i} + b^{n+1}.
\]

Hence we may apply Lemma 2 to conclude that

\[
   (a + b)^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} a^i b^{n+1-i}.
\]

This shows that the statement is true for \( n + 1 \), and so by induction it is true for all \( n \). q.e.d.

There is another form of the induction principle we shall also use, as illustrated in the next Proposition:

**Proposition 1** Suppose given a sequence of statements \( S(n) \), one for each \( n \in \mathbb{N} \), and that

1. \( S(1) \) is true.
2. Whenever \( S(k) \) is true for \( k \leq n \) then also \( S(n+1) \) is true.

Then \( S(n) \) is true for all \( n \in \mathbb{N} \).

**Proof:** Let \( T(n) \) be the statement: \( S(k) \) is true for all \( k \leq n \).

Since \( S(1) \) is true, so is \( T(1) \).

Suppose by induction that \( T(n) \) is true.

Then \( S(k) \) is true for all \( k \leq n \).

Therefore by hypothesis, \( S(n+1) \) is true.

Since \( S(k) \) is true for \( k \leq n \) as well, it is true for \( k \leq n + 1 \).

Thus \( T(n+1) \) is true.

Now it follows from the induction principle that \( T(n) \) is true for all \( n \).

Therefore \( S(n) \) is true for all \( n \). q.e.d.
Induction is also used to construct infinite sequences. This method is based on the

Construction by Induction Principle To construct a sequence \((x_n)_{n \geq 1}\) it is sufficient to

1. First, construct \(x_1\).
2. Then, assuming that \(x_i\) has been constructed for \(i \leq n\), give some explicit construction for \(x_{n+1}\).

The induction construction principle then states that this constructs a specific infinite sequence \((x_n)_{n \geq 1}\) defined for all \(n\).

Example 9 Construction by induction

1. An infinite sequence \((x_n)_{n \geq 1}\) is constructed by induction as follows:
   
   (a) Set \(x_1 = 2\).
   
   (b) Assume \(x_i\) is constructed for \(i \leq n\) and set

   \[
   x_{n+1} = \sum_{i=1}^{n} (-1)^i x_i.
   \]

2. An infinite sequence \((x_n)_{n \geq 1}\) satisfying \(x_{n+1} > x_n^2\) is constructed by induction as follows:
   
   (a) Set \(x_1 = 1\)
   
   (b) Assume \(x_i\) is constructed for \(i \leq n\) and set

   \[
   x_{n+1} = x_n^2 + 1.
   \]

3. An infinite sequence of natural numbers \((a_k)_{k \geq 1}\) is constructed by induction by setting

   \[
   a_1 = 1, \quad \text{and} \quad a_{k+1} = \begin{cases} 
   \frac{\sum_{i=k}^{\infty} a_i}{2}, & \text{if } \sum_{i=k}^{\infty} a_i \text{ is even;} \\
   k, & \text{if } \sum_{i=k}^{\infty} a_i \text{ is odd.}
   \end{cases}
   \]

Exercise 3 Induction

1. Show by induction that \(\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\).
2. Show by induction that \(\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\).
3. Use the Binomial theorem to show that if \( p, k \in \mathbb{N} \) then
\[
\left(\frac{p+1}{p}\right)^k > \frac{k}{p}.
\]
Conclude that
\[
\left(\frac{p}{p+1}\right)^k < \frac{p}{k}.
\]

4. Show by induction on \( n \in \mathbb{N} \) that the product of \( n \) odd natural numbers is odd.

5. Let \( p_1 < ... < p_n < ... \) be an infinite sequence of prime numbers with \( p_1 = 5 \). Show by induction that \( p_n > 2n \).

6. Construct by induction an infinite sequence of natural numbers \((x_n)_{n \geq 1}\) such that for \( n \geq 2 \), \( x_n > \sum_{i=1}^{n-1} x_i \).

7. Construct by induction an infinite sequence of rational numbers \((a_n)_{n \geq 1}\) such that \( a_1 = 0 \) and for all \( n, a_n < a_{n+1} \), and \( \sum_{i=1}^{n} a_i < 1 \).

2.6 Proof by Counterexamples

Counterexamples are an important way to show that a statement is false. Many mathematical statements have the form

"If A is true then B is true".

To show that this assertion is false it is enough to find a simple example in which A is true but B is not true. Such an example is called a \textbf{counterexample}.

\textbf{Example 10 Counterexamples}

1. To prove that the statement \textbf{"all natural numbers are prime"} is false you only have to find a \textit{single} example (eg. \( 6 = 2 \times 3 \)).

2. The statement that all students in Math 310 are seniors false: there are several counterexamples in each class.

3. There is a much deeper example from the theory of equations. You learned in school that the quadratic equation
\[
ax^2 + bx + c = 0
\]
has two solutions, \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \). One might ask whether the solutions to higher order equations can also be expressed in formulas that only use \( n^{th} \) roots, for some natural numbers \( n \), and indeed this is true for cubic and quartic equations.
However, it is not true for all fifth degree equations, as was shown by Abel and Ruffini in 1824. The easiest way to show this is by an explicit counterexample and a very simple one was proved about a century ago by a famous algebraist, Emil Artin, who showed that roots of the equation \(x^5 - x - 1 = 0\) cannot be expressed in this way.

**Exercise 4 Counterexamples**

1. Show that the following statement is false: If \(a, b \in \mathbb{N}\) then \(a^3 + b^3\) is the cube of a natural number.

2. Prove that the sum of an odd number of natural numbers is odd if each of the natural numbers is odd. What is the converse statement? Show by a counterexample that the converse is false.

3. What is the converse to the statement: "Suppose \(n\) is a natural number. If \(n\) is an odd natural number then \(n^2\) is odd". Decide if it is true or false and prove your answer.

4. Prove or disprove the following statement: For every \(x \in \mathbb{Q}\) there is a unique \(n \in \mathbb{N}\) which is the closest natural number to \(x\).

5. Construct an example to show that if in each of two school classes the average GPA of the boys is bigger than that of the girls it may not be the case that when the classes are combined this is still true: i.e., in the combined classes the average GPA of the girls may be bigger than that of the boys.

6. Show by counterexample that the following statement is false: "For every \(p, q \in \mathbb{N}\), \(p^2 + q^2 > (p + q)^2 - 50p - 51q\).

7. Show by counterexample that the following statement is false: "If \(t, u, s \in \mathbb{N}\) satisfy \(s > 39, t > s + 5, \) and \(u > 33\), then \(t + u > 82\).

**2.7 The Literature of Mathematics**

Mathematicians have built up a body of knowledge over several thousand years, expressed in theorems and examples, all validated by rigorous proofs, and that once validated are then true forever. This is the literature of mathematics. Each new piece of knowledge depends on what came before, and sometimes it takes many decades for little pieces to fit together to establish some remarkable new phenomenon.

The classical example is 'Fermat’s last theorem'! Fermat was a 17th century number theorist, and after he died in 1637 the following statement was found written in the margin of one of his books: "I have found the most wonderful
result, but the margin is too small for me to write down the proof.” The simple assertion was this: If $a, b, c, n$ are natural numbers all greater than 1, and if

$$a^n + b^n = c^n,$$

then $n = 2$.

In the following three hundred years the search for a proof was one of the ‘holy grails’ of mathematics. Much of our knowledge in number theory and geometry was developed in the process. And finally in 1995 Andrew Wiles (Princeton) published a proof using many of the results which had been established over the preceding centuries.

Some of you may wonder about the relation of mathematics to the physical world we live in. Now mathematical knowledge itself is validated just by proofs, and so its correctness stands for all time and does not depend in any way on our physical experience. Nonetheless, much of mathematics is inspired by that physical experience, mathematical knowledge is regularly applied in almost every branch of science and engineering, and indeed many mathematicians focus on building mathematical models and computational algorithms that directly reflect/compute physical phenomena. As a Nobel Laureate in Physics, Eugene Wigner, wrote in 1960 in a paper entitled *The Unreasonable Effectiveness of Mathematics* “…the mathematical formulation of the physicist’s often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena”.
Chapter 3

Basic Set Theory

3.1 Sets

Sets, and maps between sets, are part of the basic language of Mathematics in many areas, including algebra, analysis, logic and topology. In particular, the material in this chapter will be used throughout the rest of this course!

Definition 5 Sets

1. As stated in Section 2.1, a set is any explicitly specified collection of objects, abstract or concrete. If $S$ is a set, we sometimes write

   $$S = \{\cdots\},$$

   where the elements of the set are listed between the parentheses.

2. The 'objects' are called the elements of the set.

3. If $x$ is an element in a set $S$ we say $x$ belongs to $S$, and denote this by $x \in S$.

4. A finite set is a set with only finitely many elements. In this case $|S|$ denotes the number of elements in $S$.

5. A set which is not finite is called infinite.

6. The empty set is the set with no elements. It is denoted by $\emptyset$. 


Note: In the exercises, in order to define a set, you must specify its elements.

Example 11  Sets

1. The set $S = \{1, 2, 3\}$ has three elements: 1, 2, and 3 and so $|S| = 3$.

2. The empty set $\emptyset$ has no elements and so $|\emptyset| = 0$.

3. Each of $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are infinite sets.

4. The unit interval of all real numbers $x$ satisfying $0 \leq x \leq 1$ is an infinite set.

5. The desks in this classroom are a finite set.

6. The real numbers whose squares are natural numbers are a set.

3.2 Operations with sets

Definition 6  Subsets

A subset of a set $S$ is a set $W$ all of whose elements are elements of $S$. We denote this by $W \subseteq S$ and we often write

$$W = \{ x \in S \mid \cdots \},$$

where $\cdots$ specifies which elements of $S$ are in $W$.

Example 12  Subsets

1. The set $S$ of even integers is the subset of $\mathbb{Z}$ given by

$$S = \{ n \in \mathbb{Z} \mid n \text{ is divisible by } 2 \}.$$

It is not a finite set.

2. $\mathbb{N}$ is the subset of $\mathbb{Z}$ defined by

$$\mathbb{N} = \{ n \in \mathbb{Z} \mid n > 0 \}.$$

3. $\mathbb{Q}$ is the subset of $\mathbb{R}$ given by

$$\mathbb{Q} = \{ x \in \mathbb{R} \mid x = p/q \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \}.$$

4. The empty set is a subset of every set.

5. Let $L$ be the set whose elements are all the lines in the plane. Thus a line in the plane is a subset of the plane, and is also an element in $L$. 

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6. The vertical lines in the plane are a subset of $L$.

**Definition 7 Unions and intersections**

Suppose $S_{\alpha}$ is a collection of sets. Then

1. The **union** of the $S_{\alpha}$ is the set

$$\bigcup_{\alpha} S_{\alpha} = \{x \mid x \in S_{\alpha} \text{ for some } S_{\alpha}\}.$$  

In words: $\bigcup_{\alpha} S_{\alpha}$ is the set whose elements are those elements which belong to **at least one** of the $S_{\alpha}$.

2. The **intersection** of the $S_{\alpha}$ is the set

$$\bigcap_{\alpha} S_{\alpha} = \{x \mid x \in S_{\alpha} \text{ for each } \alpha\}.$$  

In words, $\bigcap_{\alpha} S_{\alpha}$ is the set whose elements are those elements which belong to each of the $S_{\alpha}$.

3. Two sets $S$ and $T$ are **disjoint** if $S \cap T = \emptyset$ Thus $S$ and $T$ are disjoint if the have **no** elements in common.

**Example 13 Unions and intersections**

1. The union of the sets $\{1,2,3\}$ and $\{2,4,8\}$ is $\{1,2,3,4,8\}$. The original two sets each have three elements, the union has five elements, and the intersection has one element.

2. The union of the sets $\{1,2\}$, $\{2,3\}$ and $\{3,4\}$ is the set $\{1,2,3,4\}$. The intersection of the first three sets is $\emptyset$.

3. Let $S$ be the set of even integers and let $T$ be the set of odd integers. Then

$$S \cup T = \mathbb{Z} \quad \text{and} \quad S \cap T = \emptyset.$$  

4. For each $n \in \mathbb{N}$ let $S_n \subset \mathbb{N}$ be the subset of all natural numbers except $n$. Then

(a) $$\bigcup_{n} S_n = \mathbb{N},$$  

since every natural number is in some (in fact in almost all) $S_n$.

(b) $$\bigcap_{n} S_n = \emptyset,$$

because any natural number $k$ is not $S_k$ and so is not in every $S_n$.  

Definition 8  **Products**

The **product** of two sets $S$ and $T$ is the set $S \times T$ whose elements are the ordered pairs $(x, y)$ with $x \in S$ and $y \in T$. This is written as

$$S \times T = \{(x, y) \mid x \in S \text{ and } y \in T\}.$$  

**Example 14 Products**

1. The product of the sets $S = \{2, 17, 99\}$ and $T = \{5, \sqrt{2}\}$ is the set

   $$S \times T = \{(2, 5), (17, 5), (99, 5), (2, \sqrt{2}), (17, \sqrt{2}), (99, \sqrt{2})\}.$$  

   Thus in this example, $|S| = 3$, $|T| = 2$ and $|S \times T| = 6$.

2. The product $\mathbb{R} \times \mathbb{R}$ is the set $\mathbb{R}^2$, of points in the plane:

   $$\mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}.$$  

3. The product of $\mathbb{Q}$ with $\mathbb{R}$ is the set $\{(a, y) \mid a \in \mathbb{Q}, y \in \mathbb{R}\}$ of points in the plane whose $x$-coordinate is rational. It is a subset of $\mathbb{R}^2$.

4. The set $\mathbb{Q}$ is the set of real numbers of the form $p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. This is **not** the product

   $$\mathbb{Z} \times \mathbb{N} = \{(p, q) \mid p \in \mathbb{Z}, q \in \mathbb{N}\},$$

   because $(2, 1)$ and $(4, 2)$ are different elements in the product but $2/1 = 4/2$.

Definition 9  **Power set** $P(S)$ is the set whose elements are all the subsets of $S$.

**Example 15 Power sets**

1. For any set $S$, $S \in P(S)$ and $\phi \in P(S)$ because $S$ and $\phi$ are subsets of $S$.

2. The power set $P(\{1, 2\})$ is given by

   $$P(\{1, 2\}) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}.$$  

3. The power set $P(\phi)$ of $\phi$ is the set

   $$P(\phi) = \{\phi\}.$$  

**Exercise 5 Sets**

1. Let $S$ be the set of all even integers, and let $T$ be the set of all integers divisible by 3.
(a) What is $S \cap T$?
(b) Let $W$ be the set of integers which are not in $S \cup T$. Is $W$ infinite or finite?

2. Suppose $S_n = [n, n + 1], n \in \mathbb{N}$. Find $\bigcup S_n$ and $\bigcap S_n$.

3. Suppose $S_n = [1/n, 1], n \in \mathbb{N}$. Find $\bigcup S_n$ and $\bigcap S_n$.

4. Suppose $S$ and $T$ are sets and $|S| = 3$ and $|T| = 4$
   (a) What is $|S \times T|$?
   (b) What is $|P(S)|$?

5. Suppose $S$ and $T$ are sets.
   (a) If $S$ and $T$ are disjoint sets show that for any set $W$, $S \times W$ and $T \times W$ are disjoint.
   (b) If $S$ and $T$ are disjoint sets for which sets $W$ are $S \cup W$ and $T \cup W$ disjoint?
   (c) If $S$ and $T$ are disjoint sets show that $P(S) \cap P(T) = \emptyset$.

6. If $S$ and $T$ are finite sets show that
   \[ |S \cup T| + |S \cap T| = |S| + |T|. \]

7. Let $S$ be a finite set and suppose $x \notin S$. Show that
   \[ |P(S \cup \{x\})| = 2|P(S)| \]

8. Use the previous problem and induction to prove that if $S$ is a finite set then $|P(S)| = 2^{|S|}$.

3.3 Maps between Sets

**Definition 10** Maps

A map $\varphi$ from a set $S$ to a set $T$ consists of three things:

1. A set $S$, called the **domain** of the map;
2. A set $T$, called the **target** of the map; and
3. A **rule** which assigns to each element $x \in S$ a single specified element $y \in T$. The element $y$ is called the **image** of $x$ and is denoted by $y = \varphi(x)$.

This map is often denoted by
\[ \varphi : S \rightarrow T. \]
Definition 11  More on maps

1. If \( \varphi : S \rightarrow T \) is a map between sets then the image of \( \varphi \) is the subset \( \text{Im} \varphi \subset T \) defined by
\[
\text{Im} \varphi = \{ \varphi(x) \mid x \in S \}.
\]

**NOTE:** The image of a map is a subset of the target, but it may not be all of the target!

2. If \( \varphi : S \rightarrow T \) is a map, and if \( U \subset T \) is a subset then we write
\[
\varphi(U) = \{ y \in T \mid y = \varphi(x) \text{ for some } x \in U \}.
\]

In particular, \( \text{Im} \varphi = \varphi(S) \).

3. If \( S \) and \( T \) are sets, then the collection of all maps from \( S \) to \( T \) is a set.

It is denoted by \( T^S \):
\[
T^S = \{ \varphi \mid \varphi \text{ is a map from } S \text{ to } T \}.
\]

Note:

1. To define a map we must specify **three** things: the domain, the target, and the rule - see Example 16, below.

2. **Do not confuse** a set, \( S \) with a map, \( \varphi \).
   (a) A **set** is a collection of objects.
   (b) A **map**, \( \varphi : S \rightarrow T \) is a rule connecting each element \( x \in S \) to a single element \( \varphi(x) \in T \).

Example 16 Maps

1. A map \( \varphi : \mathbb{Z} \rightarrow \mathbb{Z} \) is defined by
\[
\varphi(a) = \begin{cases} 
-a, & a \leq 0 \\
\frac{a}{2}, & a \in \mathbb{N} \text{ is even} \\
0, & a \in \mathbb{N} \text{ is odd.}
\end{cases}
\]
Maps are often described using the notation of this example.

2. **Non map vs map**
   (a) The "rule" which assigns
   \[
   1 :\rightarrow 1 \text{ and } 2
   \]
   and
   \[
   2 :\rightarrow 2
   \]
   is **not a map** from \( \{1, 2\} \) to \( \{1, 2\} \) because two different elements are assigned to 1.
(b) However, the rule which assigns 1 and 2 to 1 is a map.

3. A map $\varphi : \mathbb{N} \to \mathbb{N}$ is defined by

$$\varphi(n) = 2n.$$ 

Here the target is $\mathbb{N}$, but the image is the subset of even natural numbers.

4. Three maps are defined as follows:

(a) $\varphi : \mathbb{N} \to \mathbb{N}$; $\varphi(n) = n^2 + 1$.

(b) $\psi : \mathbb{Z} \to \mathbb{N}$; $\psi(n) = n^2 + 1$.

(c) $\chi : \mathbb{N} \to \mathbb{Z}$; $\chi(n) = n^2 + 1$.

These three maps are all different even though the "formula" is the same, because the two sets are different in each case.

5. A map $f$ from $\mathbb{R}$ to $\mathbb{R}$ is defined by $f(x) = x^2$, and a map $g$ from $\mathbb{R}$ to the set $T$ of all non-negative real numbers defined by $g(x) = x^2$. Note: as above, $f$ and $g$ are different maps even though the formula is the same.

6. Suppose $S$ is the set of students enrolled in the University of Maryland on Oct. 1, 2016. Then a map $\varphi : S \to \mathbb{N}$ is defined by the rule which assigns to each student the least number of inches which is greater than or equal to their height.

7. $S$ is the set of students enrolled in the University of Maryland on Oct. 1, 2016, $C$ is the set of all countries that existed prior to Oct. 1, 2016, and $\varphi$ is the rule which assigns to each student the country in which they were born.

8. $\psi : \mathbb{N} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ assigns to each $n \in \mathbb{N}$ the $n$th integer in the decimal expansion of $\pi$.

Remark: A map from $\mathbb{R}$ to $\mathbb{R}$ is often called a real valued function, and the formulas you are used to from calculus usually define real valued functions on $\mathbb{R}$. Polynomials defined next are an important example of such functions. However, not every real valued function has a formula.
Definition 12 (Polynomials)

A polynomial of degree $n$ ($n$ an integer $\geq 0$) is a map $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = \sum_{k=0}^{n} a_k x^k,$$

where each $a_k$ is a fixed real number and $a_n \neq 0$.

Definition 13 Composites

If $\varphi : \mathbb{S} \to \mathbb{T}$ and $\psi : \mathbb{T} \to \mathbb{W}$ are maps then their composite is the map $\psi \circ \varphi : \mathbb{S} \to \mathbb{W}$ defined by

$$(\psi \circ \varphi)(x) = \psi(\varphi(x)), \quad x \in \mathbb{S}.$$ 

Important Note: Composites can be defined more generally, but in this course we shall only consider the situation described above!

Example 17 Composites

1. The composite $g \circ f$ of a map $f : \mathbb{N} \to \mathbb{R}$ and a map $g : \mathbb{N} \to \mathbb{R}$ is not defined because the domain of $g$ is not the target of $f$.

2. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = x^2$ and $g(x) = x + 1$ then

$$(g \circ f)(x) = x^2 + 1 \text{ and } (f \circ g)(x) = (x + 1)^2.$$ 

3. Given $a \in \mathbb{Q}$ we may write $a = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. However, $p, q$ are not determined by $a$, since if $a = p/q$ then also $a = 2p/2q$. But we can define maps

$$f : \mathbb{Q} \to \mathbb{Z} \quad \text{and} \quad g : \mathbb{Q} \to \mathbb{N}$$

by setting $g(a)$ to be the least possible $q$ and $f(a)$ to be the corresponding $p$. Let $h : \mathbb{Z} \subset \mathbb{Q}$ be the inclusion. Then

$$(f \circ h)(p) = p, \; p \in \mathbb{Z}.$$ 

3.4 Onto, 1-1, and 1-1 correspondences

There are three very important classes of maps:

Definition 14

1. A map $\varphi : \mathbb{S} \to \mathbb{T}$ is onto if $\text{Im} \varphi = \mathbb{T}$. Thus $\varphi$ is onto if and only if every element $y$ can be written in the form $y = \varphi(x)$ for at least one $x \in \mathbb{S}$. 

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2. A map $\varphi : S \to T$ is 1-1 if it maps different elements in $S$ to different elements in $T$. Thus, the map $\varphi$ is 1-1 if and only if for all $x_1, x_2 \in S$, $x_1 \neq x_2 \Rightarrow \varphi(x_1) \neq \varphi(x_2)$.

3. A map $\varphi : S \to T$ is a 1-1 correspondence (also called a bijection) if and only if it is both onto and 1-1.

Caution: Do not confuse the definition of a map with the definition of a 1-1 map:

1. A map $\varphi : S \to T$ associates to each $x \in S$ a single element $\varphi(x) \in T$.

2. A map $\varphi : S \to T$ is 1-1 if whenever $x_1$ and $x_2$ are different elements in $S$ then the elements $\varphi(x_1)$ and $\varphi(x_2)$ in $T$ are also different.

In other words:

1. If $\varphi$ is a map from $S$ to $T$, then for each $x \in S$ there is a single $y \in T$ such that $\varphi$ maps $x$ to $y$.

2. If $\varphi$ is a 1-1 map from $S$ to $T$, then
   (a) for each $x \in S$ there is a single $y \in T$ such that $\varphi$ maps $x$ to $y$ (because $\varphi$ is a map)
   AND
   (b) for each $y$ in the image of $\varphi$ there is a single $x$ which is mapped to $y$.

Important Remark: A map $\varphi : S \to T$ is a 1-1 correspondence if and only if:

1. $T = \text{Im} \varphi$ (\varphi\text{ is onto}) and also
2. For every $y \in \text{Im} \varphi$ there is a single $x$ such that $\varphi(x) = y$. ($\varphi$ is 1 – 1.)

Putting these two statements together we get

$\varphi$ is a 1-1 correspondence

$\Leftrightarrow$ for each $y \in T$ there is a single $x \in S$ which is mapped to $y$.

Example 18 1. In the third example in Example 16, $\varphi$ is 1-1 but not onto, $\psi$ is neither 1-1 or onto, and $\chi$ is 1-1 but not onto.
2. The map $\varphi : \mathbb{N} \to \mathbb{N}$ given by

$$\varphi(m) = \begin{cases} 
1, & m = 1, 2, \\
m - 1, & m \geq 3
\end{cases}$$

is onto. It is not 1-1 because both 1 and 2 are mapped to 1.

3. The map $\varphi : \mathbb{Z} \to \mathbb{N}$ given by

$$\varphi(k) = \begin{cases} 
-2(k-1), & k \leq 0, \\
2k-1, & k \geq 1
\end{cases}$$

is 1-1 and onto. Thus it is a 1-1 correspondence.

4. Let $f : \{1, 2, 3, 4\} \to \{a, b, c, d\}$ be defined by $f(1) = c, f(2) = a, f(3) = d,$ and $f(4) = b.$ and let $h : \{a, b, c, d\} \to \{a, b, c, d\}$ be defined by $h(a) = b, h(b) = c, h(c) = d, h(d) = d.$ Then

(a) $f \circ h$ is not defined.

(b) $h$ is not onto and is not 1-1.

(c) $h \circ f$ is given by $(h \circ f)(1) = d, (h \circ f)(2) = b, (h \circ f)(3) = d,$ and $(h \circ f)(4) = c.$

5. The map $\varphi : \mathbb{N} \to \mathbb{N}$ defined by

$$\varphi(n) = n + 1$$

is 1-1 but not onto, since 1 is not in the image.

6. The map $\psi : \mathbb{N} \to \mathbb{N}$ defined by

$$\psi(1) = \psi(2) = 1 \text{ and } \psi(n) = n - 1 \text{ if } n \geq 3$$

is onto but not 1-1.

**Exercise 6 Maps**

1. Suppose $f : S \to T$ is a map and $U$ and $W$ are subsets of $S$.

   (a) If $U \cap W = \emptyset$ does it follow that $f(U) \cap f(W) = \emptyset$?

   (b) If $f(U) \cap f(W) = \emptyset$ does it follow that $U \cap W = \emptyset$?

2. List all the maps $\varphi$ from $S = \{1, 2\}$ to $T = \{-1, -2\}$ such that $\text{Im} \varphi = T$.

3. Construct sets $S, T, U, W$ and maps $\varphi : S \to T$ and $\psi : U \to W$ such that a composite $\psi \circ \varphi : S \to W$ is not defined but the composite $\varphi \circ \psi : U \to T$ is defined.

4. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \{x \in \mathbb{R} \mid x \geq 0\} \to \mathbb{R}$ are defined by $f(x) = x^2$ and $g(x) = \sqrt{x}$ are $f \circ g$ and $g \circ f$ defined?
5. Construct an example of maps \( f : S \to T \) and \( g : T \to S \) in which \( f \) is 1-1 and \( g \) is onto, but the composite \( g \circ f \) is neither 1-1 nor onto.

6. Construct a 1-1 correspondence from \( \mathbb{Z} \) to \( \mathbb{N} \).

7. If either \( S = \emptyset \) or \( T = \emptyset \) show that \( T^S = \emptyset \).

8. (Division Proposition for polynomials) Suppose \( f \) and \( g \) are polynomials and \( \deg g < \deg f \). Show there is a polynomial \( h \) such that either \( f = hg \) or else \( r = f - gh \) is a polynomial and \( \deg r < \deg g \).

**Definition 15** Associativity, the identity, and inverses.

1. The **identity map** of a set \( S \) is the map \( \text{id}_S : S \to S \) defined by
   \[
   \text{id}_S(x) = x, \quad x \in S.
   \]

2. If \( \varphi : S \to T \) is a 1-1 correspondence, then the **inverse** of \( \varphi \) is the map \( \varphi^{-1} : T \to S \) defined by
   \[
   \varphi^{-1}(y) = \text{the unique} \ x \in S \text{ such that} \ \varphi(x) = y.
   \]

**Proposition 2** Suppose \( \varphi : S \to T \), \( \psi : T \to W \), and \( \chi : W \to U \) are maps of sets. Then

1. For \( x \in S \),
   \[
   \text{id}_T \circ \varphi = \varphi \quad \text{and} \quad \varphi \circ \text{id}_S = \varphi.
   \]

2. (**Associativity** of composition):
   \[
   (\chi \circ \psi) \circ \varphi = \chi \circ (\psi \circ \varphi).
   \]

3. If \( \varphi \) is a 1-1 correspondence then
   \[
   \varphi^{-1} \circ \varphi = \text{id}_S \quad \text{and} \quad \varphi \circ \varphi^{-1} = \text{id}_T.
   \]

4. If there is a map \( \xi : T \to S \) satisfying
   \[
   \xi \circ \varphi = \text{id}_S \quad \text{and} \quad \varphi \circ \xi = \text{id}_T,
   \]
   then \( \varphi \) is a 1-1 correspondence and \( \xi = \varphi^{-1} \).

**Proof:**

1. For \( x \in S \),
   \[
   (\text{id}_T \circ \varphi)(x) = \text{id}_T(\varphi(x)) = \varphi(x) \quad \text{and} \quad (\varphi \circ \text{id}_S)(x) = \varphi(\text{id}_S(x)) = \varphi(x).
   \]

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2. For \( x \in S \),

\[
((\chi \circ \psi) \circ \varphi)(x) = (\chi \circ \psi)(\varphi(x)) = \chi(\psi(\varphi(x)))
\]

and

\[
(\chi \circ (\psi \circ \varphi))(x) = \chi((\psi \circ \varphi)(x)) = \chi(\psi(\varphi(x))).
\]

Thus both sides evaluated at any \( x \in S \) give \( \chi(\psi(\varphi(x))) \).

3. For \( x \in S \), \( x \) is the unique element mapped to \( \varphi(x) \) by \( \varphi \). Thus by definition

\[
\varphi^{-1}(\varphi(x)) = x = id_S(x).
\]

Moreover, if \( y \in T \) the \( y = \varphi(x) \) for a unique \( x \in S \). By definition, \( x = \varphi^{-1}(y) \). Thus

\[
\varphi(\varphi^{-1}(y)) = \varphi(x) = y = id_T(y).
\]

4. If \( y \in T \) then

\[
y = \varphi(\xi(y)) \in \text{Im} \varphi.
\]

Therefore \( \varphi \) is onto.

Moreover, if \( x_1, x_2 \in S \) and \( \varphi(x_1) = \varphi(x_2) \) then

\[
x_1 = \xi(\varphi(x_1)) = \xi(\varphi(x_2)) = x_2.
\]

Therefore \( \varphi \) is 1-1, and so it is a 1-1 correspondence.

Finally,

\[
\xi = \xi \circ id_T = \xi \circ (\varphi \circ \varphi^{-1}) = (\xi \circ \varphi) \circ \varphi^{-1} = o\varphi^{-1}.
\]

q.e.d.

**Example 19** Inverses

1. Let \( f : \{1, 2, 3, 4\} \to \{a, b, c, d\} \) be defined by \( f(1) = c, f(2) = a, f(3) = d, \) and \( f(4) = b \). Then \( f \) is a 1-1 correspondence and its inverse is given by \( f^{-1}(a) = 2, f^{-1}(b) = 4, f^{-1}(c) = 1, f^{-1}(d) = 3 \).

2. Let \( A \) be an \( n \times n \) matrix, and regard \( A \) as a map from the set of column vectors to itself by setting \( A(v) = Av \) (matrix multiplication). If \( B \) is a second \( n \times n \) matrix, then \( (A \circ B)(v) = A(B(v)) = A(Bv) = (AB)v \). Thus the map \( A \circ B \) is just multiplication by the product matrix \( AB \).

3. In the example above, suppose \( A \) is an invertible matrix with inverse matrix \( A^{-1} \). Then using elementary linear algebra it follows that multiplication by \( A \) is a bijection, and that the inverse map is multiplication by \( A^{-1} \).
4. It follows from what you learn in an elementary calculus course that the map \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x^3 \) is a 1-1 correspondence. The inverse maps any real number \( x \) to its unique cube root.

5. Let \( (x_n)_{n \geq 1} \) be a sequence of real numbers such that for all \( n, x_n < x_{n+1} \). Then define \( S \subset \mathbb{R} \) by
   \[
   S = \{ x_n \mid n \geq 1 \},
   \]
   and define \( f : \mathbb{R} \to \mathbb{R} \) by setting
   \[
   f(x) = \begin{cases} 
   x, & x \notin S \\
   x_{2k-1}, & x = x_{2k}, \\
   x_{2k}, & x = x_{2k-1}. 
   \end{cases}
   \]
   Then \( f \) is a 1-1 correspondence.

**Proposition 3** Let \( S \) be a non-void set. Then a 1-1 correspondence,
   \[
   \gamma : P(S) \to \{0,1\}^S
   \]
   is given by
   \[
   \gamma(U)(x) = \begin{cases} 
   1 & x \in U \\
   0 & x \notin U.
   \end{cases}
   \]

**Proof:** Define \( \chi : \{0,1\}^S \to P(S) \) by setting
   \[
   \chi(f) = \{ x \in S \mid f(x) = 1 \},
   \]
   where \( f : S \to \{0,1\} \) is any map. Then for \( U \subset S \),
   \[
   (\chi \circ \gamma)(U) = \{ x \in S \mid \gamma(U)(x) = 1 \} = U.
   \]
   Also, for \( f : S \to \{0,1\} \),
   \[
   (\gamma \circ \chi)(f) = \gamma(\{ x \in S \mid f(x) = 1 \}) = f.
   \]
   Therefore by Proposition 2, \( \gamma \) is a 1-1 correspondence with inverse \( \chi \). \textbf{q.e.d.}

**Theorem 3** If \( S \) is any non-void set then no map from \( S \) to \( P(S) \) is onto.

**Proof:** In view of Proposition 3 it is sufficient to prove that no map from \( S \) to \( \{0,1\}^S \) is onto. We do this by contradiction, and assume for some
   \[
   \Phi : S \to \{0,1\}^S
   \]
   that \( \Phi \) is onto.

Now by definition, for each \( x \in S \), \( \Phi(x) \) is a map from \( S \) to \( \{0,1\} \). Define
   \[
   f : S \to \{0,1\}
   \]
by setting
\[
f(x) = \begin{cases} 
1 & \Phi(x)(x) = 0 \\
0 & \Phi(x)(x) = 1.
\end{cases}
\]
We shall show that \(f\) is not in the image of \(\Phi\) and so \(\Phi\) is not onto.

By construction it is never true for any \(x \in S\) that \(\Phi(x)(x) = f(x)\). But if 
\(\Phi(y) = f\) for some \(y \in S\) then by definition,
\[
\Phi(y)(z) = f(z), \ z \in S.
\]
In particular,
\[
\Phi(y)(y) = f(y),
\]
which is impossible. Thus \(f\) is not in \(\text{Im } \Phi\) and so \(\Phi\) is not onto.
q.e.d.

**Exercise 7** Sets and maps

1. Determine which of the following is a map:
   
   (a) \(\varphi : \mathbb{N} \to \mathbb{N}\), defined by
   \[
   \varphi(x) = \begin{cases} 
1,2 & x = 1, \\
2x & x > 1.
\end{cases}
\]
   
   (b) \(\psi : \mathbb{R} \to \mathbb{R}\), defined by
   \[
   \psi(x) = \begin{cases} 
1 & x = 1, \\
2x & x \neq 1.
\end{cases}
\]
   
   (c) \(\omega : \mathbb{C} \to \mathbb{C}\) defined by:
   \[
   \omega(x) = \begin{cases} 
5 & x = 1,2,3 \\
2x & x \neq 1,2,3.
\end{cases}
\]

2. Which of the maps in Problem 1. are 1-1, onto?

3. For \(\psi\) and \(\omega\) in Problem 1. evaluate \(\psi \circ \psi\) and \(\omega \circ \omega\).

4. If \(S\) and \(T\) are finite sets show that \(|S \times T| = |S||T|\).

5. If \(S\) is a finite set show that \(|P(S)| = 2^{|S|}\).

6. Prove that:
   
   (a) for any set \(S\), \(\text{id}_S\) is a bijection.
   
   (b) The composite of onto set maps \(\varphi\) and \(\psi\) is onto.
(c) The composite of 1-1 set maps is 1-1.

(d) If $\varphi : S \rightarrow T$ and $\psi : T \rightarrow W$ are 1-1 correspondences show that the inverse of $\varphi$ and also $\psi \circ \varphi$ are 1-1 correspondences.

7. Suppose $\varphi : S \rightarrow T$ is a map.

(a) If $T$ is finite and the map is 1-1 show that $S$ is finite and that $|S| \leq |T|$.

(b) If $S$ is finite and the map is onto show that $|T|$ is finite and that $|S| \geq |T|$.

8. Suppose $\varphi : S \rightarrow T$ is a map and that $S$ and $T$ are finite sets with the same number of elements. Show that the following three conditions are equivalent:

(a) $\varphi$ is a 1-1 correspondence.

(b) $\varphi$ is onto.

(c) $\varphi$ is 1-1.

9. Construct two maps $\alpha$ and $\beta$, both from the same set $S$ to the same set $T$, and such that $S$ and $T$ are different sets and $\alpha$ is 1-1 but not onto and $\beta$ is onto but not 1-1.

10. If $S$ and $T$ are finite sets show that there are $|T|^{|S|}$ elements in the set of maps from $S$ to $T$.

11. How many elements are there in the set of 1-1 correspondences of a finite set $S$ to itself?

3.5 Equivalence Relations

A relation between two sets, $S$ and $T$ is a generalization of the idea of a map:

Definition 16 A relation, $R$, between a set $S$ and a set $T$ is a subset $R \subset S \times T$ of the product of $S$ with $T$.

If $(x, y) \in R$ we say that $y$ is related to $x$ by the relation $R$, and we write $x R y$.

If $S = T$ we say that $R$ is a relation in $S$.

Example 20 Suppose $\varphi : S \rightarrow T$ is a map. Then the relation

$$\{(x, \varphi(x)) \mid x \in S\} \subset S \times T$$

is called the relation of the map $\varphi$. 
Thus relations corresponding to set maps are those subsets of $S \times T$ that satisfy the following property: every $x \in S$ is related to a single $y \in T$.

By contrast, any subset $R \subseteq S \times T$ is a relation between $S$ and $T$.

**Exercise 8 Relations**

1. Identify whether which of the following subsets $R \subseteq S \times T$ is a relation determined by a map:

   (a) $S$ is all of $\mathbb{R}$ and $T$ is all of $\mathbb{Z}$, and $R = \{(2, -3), (3, -4), (4, -5)\}$.

   (b) $S = \mathbb{N}$, $T = \mathbb{R}$ and $R = \{(n, 1/n) | n \in \mathbb{N}\}$.

   (c) $S = \mathbb{N}$, $T = \mathbb{N}$ and $R = \{(n^2, 1/n) | n \in \mathbb{N}\}$.

2. Show that the relation $\{(x^2, x) | x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ is not the relation of a map.

3. Which are the integers $n \in \mathbb{N}$ such that the relation $\{(x^n, x) | x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ is the relation of a map from $\mathbb{R}$ to $\mathbb{R}$?

4. If $S$ and $T$ are finite sets, how many relations are there between $S$ and $T$? Compare this with the number of maps from $S$ to $T$ - see Exercise 5.

Perhaps you have heard the expression,

**You can’t see the forest for the trees.**

This speaker is accusing the audience that they are so bound up in the details of the trees that they don’t come to grips with the more global properties of the forests. And in fact, sometimes instead of looking at individual trees, we want to look at the forests as individuals. Similarly, instead of considering individual people we might want to talk about cities, which are collections of people. We might want to describe the properties of species, which are collections of animals, rather than the distinctions between individual animals.

This leads to a very important mathematical idea, which we formalize in the following way:

**Definition 17** A partition of a set $S$ is a family of non-empty subsets $S_i \subset S$ such that every element $x \in S$ belongs to exactly one subset $S_i$.

Every partition $\{S_i\}$ of a set $S$ determines a specific relation in $S \times S$; namely, we set $xRy$ if and only if $x$ and $y$ are in the same subset $S_i$. This is called the relation of the partition.

**Example 21**

1. The single set $S$ is a partition of a non-empty set $S$. The corresponding relation is $xRy$ for every $x, y \in S$. Thus $R = S \times S$.  

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2. The family of subsets $S_x, x \in S$ of a non-empty set $S$ is a partition of $S$, since every $x$ belongs to a single $S_x$. The corresponding relation is $xRy$ if and only if $x = y$.

3. A partition of the set, $S$, of sophomore students at the University of Maryland is defined by: For each possible GPA, $\alpha$, $S_{\alpha}$ is the set of students whose GPA is $\alpha$.

**Lemma 3** The relation of a partition $\{S_i\}$ of a set $S$ satisfies the following properties:

1. For every $x \in S$, $xRx$. (The relation is reflexive.)

2. If $xRy$ then also $yRx$. (The relation is symmetric.)

3. If $xRy$ and $yRz$ then $xRz$. (The relation is transitive.)

**Proof:**

1. The relation is reflexive because $x$ and $x$ belong to the same $S_i$.

2. If $xRy$ then by definition $x$ and $y$ are in the same $S_i$ and so $yRx$. Therefore the relation is symmetric.

3. We have to show that if $xRy$ and $yRz$ then $xRz$.

By hypothesis, $x$ belongs to exactly one $S_i$.
Since $xRy$, $y$ must also be in that $S_i$.
Since $yRz$, $z$ must also be in that $S_i$.
Thus $y$ and $z$ are in the same $S_i$.
Therefore by definition, $xRz$ and the relation is transitive.

**q.e.d.**

**Definition 18** An **equivalence relation** in a set $S$ is a reflexive, symmetric, and transitive relation. Equivalence relations are denoted by $x \sim y$.

Lemma 3 states that the relation of a partition is an equivalence relation. Conversely we have

**Proposition 4** Every equivalence relation, $\sim$, in a set $S$ is the relation of a unique partition of $S$.

**Proof:** Fix an equivalence relation, $\sim$, in $S$.

For each $x \in S$ define a subset $S(x) \subset S$ by

$$S(x) = \{y \in S \mid y \sim x\}.$$
Now let \( \{S_i\} \) denote the family of distinct subsets of \( S \) such that each \( S_i = S(x) \) for some \( x \in S \).

We will prove three things:

1. This is a partition of \( S \).
2. \( \sim \) is the relation of the partition.
3. This is the only partition with \( \sim \) as its relation.

**Step 1.** This is a partition of \( S \).

By reflexivity each \( x \in S(x) \) and so the \( S_i \) are not empty and every \( x \) in \( S \) belongs to some \( S_i \).

Thus to show this is a partition we have to show that any element of \( S \) can belong to only one subset \( S_i \).

In other words we have to show that if \( y \in S(x) \) and \( y \in S(z) \) then \( S(x) = S(z) \).

We first prove that for any \( x \in S \), if \( u \in S(x) \) then

\[
S(u) \subseteq S(x).
\]

In fact, since \( u \in S(x) \) we have \( u \sim x \), and if \( v \in S(u) \) we have \( v \sim u \).

Thus \( v \sim u \sim x \), and since the relation is transitive, \( v \sim x \).

It follows that \( v \in S(x) \); i.e. \( S(u) \subseteq S(x) \).

Next we prove that if \( u \in S(x) \) then

\[
S(x) \subseteq S(u).
\]

In fact, since \( u \sim x \) and the relation is symmetric, \( x \sim u \).

Therefore \( x \in S(u) \) and by what we have just shown it follows that

\[
S(x) \subseteq S(u).
\]

Altogether we have proved that if \( u \in S(x) \) then \( S(x) = S(u) \).

Finally, if \( y \in S(x) \) and \( y \in S(z) \), then by the equality we just proved

\[
S(y) = S(x) = S(z).
\]

Therefore \( \{S_i\} \) is a partition and Step 1 is proved.

**Step 2.** The equivalence relation \( \sim \) is the relation of the partition.
Denote the relation of the partition by $R$, so that $xRy$ if and only if $x$ and $y$ are in the same subset $S_i$ of the partition.

Then by definition,

$$xRy \iff x, y \in S(w), \text{ some } w \in S.$$ 

But we know that if $x, y \in S(w)$ then $S(x) = S(w) = S(y)$ and so $x \sim y$. Thus $R = \sim$.

**Step 3.** The partition $\{S_i\}$ is the unique partition which has $\sim$ as its equivalence relation.

Let $\{T_\alpha\}$ be a partition of $S$ with $\sim$ as its equivalence relation.

If $x \in S$ then the subset $T_\alpha$ in the partition containing $x$ will satisfy

$$y \sim x \text{ if and only if } y \in T_\alpha.$$ 

Thus $T_\alpha = S(x)$.

It follows that every $T_\alpha$ is one of the $S(x)$.

Since every element of $S$ is in some $T_\alpha$ it follows that every $S(x)$ is one of the $T_\alpha$.

Therefore the partition $\{T_\alpha\}$ is the partition $\{S_i\}$.  \textbf{q.e.d.}

**Definition 19** If $\sim$ is an equivalence relation in a set $S$ then the elements $S_i$ of the corresponding partition are called the **equivalence classes** of the relation.

**Exercise 9** Equivalence relations

1. Provide a complete proof for the statement in Example 6.3.

2. Two UM students are related if they both take at least one class in common. Is this an equivalence relation?

3. Two UM students are related if they take all their classes in common. Is this an equivalence relation?

4. Suppose $f : S \to T$ is a map between two sets. Say $x$ and $y$ in $S$ are related if $f(x) = f(y)$. Is this an equivalence relation?

5. Suppose $S = \bigcup_{i=1}^m S_i$ is a partition of a set with $n$ elements for some natural number $n$.

   (a) If each $S_i$ has the same number of elements, $k$, show that $k$ divides $n$ and that $n/k = m$. 

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(b) If each $S_i$ has $i$ elements find $n$.

(c) If each $S_i$ has $i^3$ elements, find $n$.

(d) If each $S_i$ has an odd number of elements show that $m$ is even if and only if $n$ is even.

6. Recall that $P(Q)$ denotes the set of all subsets of $Q$. Thus the elements of $P(Q)$ are the subsets $S \subseteq Q$. Define a relation

$$\alpha \subseteq P(Q) \times P(Q)$$

by setting $S \alpha T$ if

(a) for every $a \in S$ there is some $b \in T$ such that $b \geq a$, and

(b) for every $c \in T$ there is some $d \in S$ such that $d \geq c$.

Then

(a) Show that $\alpha$ is an equivalence relation.

(b) Show that if \{ $S_i$ \} is an equivalence class of subsets then

$$\bigcup_i S_i$$

is an element in that equivalence class.

(c) Show that if $S \subseteq Q$ is not equivalent to $Q$ then for some rational number, $a$, $a > x$ for all $x \in S$.

(d) If $a \in Q$ are the sets

$$S = \{ b \in Q \mid b < a \} \text{ and } T = \{ b \in Q \mid b \leq a \}$$

equivalent?

7. Let $T^S$ denote the set of maps from a set $S$ to a set $T$. Define a relation $R \subseteq T^S \times T^S$ by setting

$$fRg$$

if $f(x) = g(x)$ except for finitely many points $x \in S$ (depending, of course, on $f$ and $g$). Show this is an equivalence relation.
Chapter 4

The Real Numbers

The bedrock of analysis is an understanding of the real numbers. Notice the difference between the rational numbers and the real numbers: we can describe the rational numbers explicitly as quotients of one integer by another. No such simple expression is available for the reals. We may have an intuitive understanding of these as the points on the real number line, as distances, or as (possibly) infinite decimals, but we cannot use “intuitive understandings” to make rigorous proofs.

We solve this problem by showing that a real number can be approximated arbitrarily well by a rational number. Explicitly:

If $x \in \mathbb{R}$ then for any $\delta > 0$ there is a rational number $a \in \mathbb{Q}$ such that $|x - a| < \delta$.

This allows us to extend properties of the rationals to the reals in a rigorous way.

Our first step is to identify the basic properties of the rationals.

4.1 Properties of the Rational Numbers

Here are the two fundamental concepts for the rationals with which we are familiar:
1. **Algebra:** The operations of addition, subtraction, multiplication and division.

2. **Order:** If \( b \) and \( a \) are rational numbers then
   \[
a < b \iff b - a = m/n \quad \text{for some} \quad m, n \in \mathbb{N}.
   \]

**Notation:** We use \( a > b \) to mean the same thing as \( b < a \).

The basic properties of the ordering in the rationals are contained in the next lemma. While they are utterly and totally familiar we shall give formal proofs to provide more examples of what a proof looks like.

**Lemma 4**

1. For each rational number \( a \in \mathbb{Q} \) exactly one of the following three possibilities is true:
   
   \begin{align*}
   & (a) \ a > 0 \\
   & (b) \ a = 0 \\
   & (c) \ a < 0
   \end{align*}

   In particular, if \( a, b \in \mathbb{Q} \) then
   \[
   b > a \iff b - a > 0.
   \]

2. **If** \( a, b, c \) **are any three rational numbers**, **then**:
   
   \begin{align*}
   & (a) \text{ If } a < b \text{ and } b < c \text{ then } a < c. \\
   & (b) \ a + b < a + c \text{ if and only if } b < c. \\
   & (c) \text{ Multiplication by a positive preserves inequalities.} \\
   & (d) \text{ Multiplication by a negative reverses inequalities.}
   \end{align*}

3. **If** \( a \in \mathbb{Q} \) **then** \( a > 0 \iff a > 1/n \) **for some** \( n \in \mathbb{N} \).

**Proof:** We prove each statement separately.

1. By definition \( a = m/n \) with \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \). Thus exactly one of the following three possibilities is true:
   
   \begin{align*}
   & (a) \ m \in \mathbb{N}, \\
   & (b) \ m = 0, \text{ or} \\
   & (c) \ -m \in \mathbb{N}.
   \end{align*}

   In the first case \( a > 0 \).
   In the second case, \( a = 0 \).
   In the third case \( a < 0 \).
2. (a) If \(a < b\) and \(b < c\) then

\[ b - a = \frac{p}{q} \text{ and } c - b = \frac{m}{n}, \]

with \(p, q, m, n \in \mathbb{N}\).

It follows that

\[ c - a = (c - b) + (b - a) = \frac{m}{n} + \frac{p}{q} > 0. \]

(b) Since \((a + c) - (a + b) = c - b\) it follows that

\[ (a + c) - (a + b) > 0 \Leftrightarrow c - b > 0. \]

(c) If \(c > 0\) and \(b - a > 0\), then for some \(p, q, m, n \in \mathbb{N}\)

\[ c = \frac{p}{q} \text{ and } b - a = \frac{m}{n}. \]

Therefore

\[ cb - ca = c(b - a) = \frac{pm}{qn} > 0. \]

(d) If \(c < 0\) and \(a < b\) then \(-c > 0\) and,

\[ -c(b - a) > 0. \]

Adding \(c(b - a)\) to both sides gives \(0 > c(b - a)\) and so

\[ ca > cb \]

3. If \(a > 0\), we have for some \(m, n \in \mathbb{N}\) that

\[ a = \frac{m}{n} > \frac{1}{n}. \]

Conversely, for any \(n \in \mathbb{N}\), \(1/n > 0\).

Thus if \(a > 1/n\) then \(a > 1/n > 0\).

q.e.d.

4.2 Introducing the Real Numbers and their Inequalities

Recall that while we may have an intuitive understanding of the real numbers as the points on the x-axis of the plane, or as distances, or as (possibly) infinite decimals, we cannot use “intuitive understandings” to make rigorous proofs. Thus we introduce the reals by listing four elementary properties as axioms, from which all the other properties will be deduced. To rigorously extend properties of the rationals to the reals we will use the fact that any real number can be
approximated by rational numbers with arbitrarily small error. Thus working with inequalities will be an important technique for the rest of this course.

Of the four basic properties we shall assume about the real numbers, the first three are immediately below. The fourth will be stated in Sec. 4.5.

**Property One:** The real numbers are a set, denoted by \( \mathbb{R} \), and containing the rational numbers.

**Property Two (Algebra):** The operations of addition, subtraction, multiplication and division are defined for real numbers, coinciding with the old operations in the rationals, and with the same properties.

**Property Three (Order):** The ordering of the rationals extends to an ordering of the real numbers such that

1. For each real number \( x \) exactly one of the following three possibilities is true:
   
   (a) \( x > 0 \), in which case we say \( x \) is **positive**.
   
   (b) \( x = 0 \).
   
   (c) \( x < 0 \), in which case we say \( x \) is **negative**.

2. \( x < y \iff y - x > 0 \).

3. If \( x < y \) and \( y < z \) then \( x < z \).

4. The product of positive real numbers is positive.

5. If \( x \in \mathbb{R} \) then there is a natural number \( m \) such that

\[
x < m.
\]

**Notation:** We shall write \( y > x \) to mean \( x < y \).

**Lemma 5**

1. If \( x, y, z \) are any three real numbers, then:
   
   (a) \( x + y < x + z \iff y < z \).
   
   (b) \( x > 0 \iff -x < 0 \). In this case \( 1/x > 0 \).
   
   (c) Multiplication by a **positive** real number preserves inequalities:

\[
z > 0 \quad \text{and} \quad x < y \Rightarrow zx < zy.
\]

(d) If \( 0 < x < y \) then \( 1/y < 1/x \).
(e) **Multiplication by a negative real number reverses inequalities:**

\[ z < 0 \quad \text{and} \quad x < y \Rightarrow zx > zy. \]

2. For each \( x \in \mathbb{R} \) there is a natural number \( n \) such that

\[ -n < x. \]

3. For each positive real number \( x \) there are natural numbers \( k, m \) such that

\[ \frac{1}{k} < x < m. \]

**Proof:** We prove each statement separately.

1. Suppose \( x, y, z \in \mathbb{R} \).

(a) Since \((x + z) - (x + y) = z - y\) it follows from Property Three that

\[ x + y < x + z \iff y < z. \]

(b) If \( 0 < x \) then \(-x + 0 < -x + x\) so that

\[ -x = -x + 0 < -x + x = 0. \]

Conversely, if \(-x < 0\), then adding \( x \) to both sides gives \( 0 < x \).

Finally, we show by contradiction that

\[ x > 0 \Rightarrow 1/x > 0. \]

In fact, suppose \( 1/x \leq 0 \). Then by what we just proved, \(-1/x \geq 0\). Since the product of positives is positive (Property 3) it would follow that

\[ -1 = (-1/x)x \geq 0, \]

which is false. It follows by contradiction that \( 1/x > 0 \).

(c) Since \( x < y \) Property 3 states that \( y - x > 0 \).

Since \( z > 0 \) and the product of positive real numbers is positive (Property 3), it follows that

\[ zy - zx = z(y - x) > 0. \]

(d) Since \( x, y > 0 \), by part (b), \( 1/x, 1/y > 0 \). Now by Property Three

\[ 1/xy = (1/x)(1/y) > o. \]

Multiplication by \( 1/xy \) therefore preserves the inequality \( x < y \) and this gives \( 1/y < 1/x \).
(e) First, if \( x < y \) then

\[
-y = x + (-x - y) < y + (-x - y) = -x,
\]

and so multiplication by \(-1\) reverses inequalities. Thus in general, if \( z < 0 \) and \( x < y \) then we have \(-z > 0\) and so by Property 3,

\[
-(zx) = (-z)x < (-z)y = -(zy).
\]

Multiplication by \(-1\) reverses this inequality and gives \( zx > zy \).

2. By Property 3 applied to \(-x\), for some \( n \in \mathbb{N} \) we have \(-x < n\). Since multiplication by \(-1\) reverses inequalities it follows that

\[
-n < x.
\]

3. Since \( x > 0 \), \( 1/x > 0 \). Thus by Property Three there are natural numbers \( k, m \in \mathbb{N} \) such that

\[
1/x < k \quad \text{and} \quad x < m.
\]

By what we proved above this implies that \( 1/k < x \) and so

\[
1/k < x < m.
\]

q.e.d.

The preceding Lemma establishes properties you have been taking for granted for years. Henceforth we will use these properties without referring to the Lemma for justification.

The next result is absolutely fundamental for this course. It states that any real number can be approximated arbitrarily well by rational numbers.

**Theorem 4 (Sandwich theorem)** Suppose \( x \in \mathbb{R} \) and \( k \in \mathbb{N} \).

1. Then there are rational numbers \( a, b \) such that

\[
a < x < b \quad \text{and} \quad b - a < 1/k.
\]

2. In particular,

\[
0 < x - a < 1/k \quad \text{and} \quad 0 < b - x < 1/k.
\]

**Proof:**

1. By Property Three and Lemma 5 there are natural numbers \( i, j \) such that

\[
-j < x < i.
\]
Set \( p = i(2k + 1) \) and \( q = j(2k + 1) \). Then we may rewrite this inequality as

\[
-\frac{q}{2k + 1} < x < \frac{p}{2k + 1}.
\]

Observe first that if \( m \in \mathbb{Z} \) and \( x < m/(2k + 1) \) then

\[
-\frac{q}{2k + 1} < x < \frac{m}{2k + 1},
\]

and so \(-q < m\).

Therefore there is a least integer \( r \) such that

\[
x < \frac{r}{2k + 1}.
\]

Then

\[
x \geq \frac{r - 1}{2k + 1} \quad \text{and so} \quad x > \frac{r - 2}{2k + 1}.
\]

Thus

\[
\frac{r - 2}{2k + 1} < x < \frac{r}{2k + 1}
\]

and

\[
\frac{r}{2k + 1} - \frac{r - 2}{2k + 1} = \frac{2}{2k + 1} < 1/k.
\]

2. This is immediate from what was just proved.

q.e.d.

**Proposition 5** If \( x < y \) are real numbers then for some \( b \in \mathbb{Q} \),

\[
x < b < y.
\]

**Proof:** By Lemma 5, for some \( k \in \mathbb{N}, 1/k < y - x \) and so \( x + 1/k < y \). Now by the Sandwich theorem, for some \( b \in \mathbb{Q} \),

\[
x < b \quad \text{and} \quad b - x < 1/k.
\]

since \( b \) was chosen so that \( x + 1/k < y \) it follows that

\[
b < x + 1/k < y.
\]

q.e.d.

**Proposition 6** If \( x > 0 \) and \( y > 1 \) are real numbers then for some \( p \in \mathbb{N} \),

\[
1/y^p < x.
\]
**Proof:** Since $y - 1 > 0$, by Lemma 5, for some $k \in \mathbb{N}, y - 1 > 1/k$. Therefore

\[ y > (1/k) + 1. \]

Now apply the Binomial theorem to obtain that for any $p \in \mathbb{N}:

\[(1/(k) + 1)^p = 1 + p/k + \sum_{i=2}^{p} \binom{p}{i} (1/k)^i > p/k.\]

Moreover, by the Difference theorem,

\[y^p - (1/k + 1)^p = (y - (1/k + 1)) \sum_{i=0}^{p-1} y^i (1/k + 1)^{p-1-i} > 0.\]

Therefore

\[y^p > (1/k + 1)^p > p/k.\]

It follows from Lemma 5 that for any $p \in \mathbb{N}$

\[1/y^p < k/p.\]

Finally, by Lemma 5, for some $n \in \mathbb{N}, x > 1/n$. Choose $p = kn$. Then

\[1/y^p < k/kn = 1/n < x.\]

q.e.d.

**Exercise 10 Inequalities**

1. Is it true that there are rational numbers $a < b$ such that for any natural number $k, b - a < 1/k$?

2. If $x, y, z, w$ are positive real numbers such that $x < y$ and $z < w$ show that $xz < yw$. State the converse and either prove it or provide a counterexample to show it is false.

3. Show that if $x < y$ are real numbers then there are infinitely many rational numbers $b$ such that $x < b < y$.

4. Suppose $c \in \mathbb{Q}$ and $y, z$ are any real numbers. Show that if $c > y + z$ then there are rational numbers $a > y$ and $b > z$ such that

\[c > a + b.\]

5. Show that the product of a positive real number with a negative real number is negative.

6. Let $x$ and $y$ be positive real numbers. Show that if $x > 1$ then (i) $xy > y$, (ii) $0 < 1/x < 1$, and (iii) $0 < y/x < y$.

7. If $a < b$ are non-negative real numbers show that for any $p \in \mathbb{N}, b^p - a^p \leq p(b - a)b^{p-1}$. (Hint: use the Difference theorem).
4.3 Absolute value

Recall that the \textbf{absolute value} of a real number \( x \) is denoted by \(|x|\) and is defined by:

\(|x| = x \) if \( x \geq 0 \) and \(|x| = -x \) if \( x < 0 \).

\textbf{Note:}

1. \(-x\) is positive when \( x \) is negative!
2. For every \( x \in \mathbb{R} \):
   \[
   |x| \geq 0 \\
   \text{and} \\
   |x| = 0 \iff x = 0.
   \]
3. \(|x|\) is the larger of \( x \) and \(-x\).
4. Thus the statement \(|x| < \varepsilon\) is the same as \( x < \varepsilon \) \textit{and} \(-x < \varepsilon\), and hence
   \[
   |x| < \varepsilon \iff -\varepsilon < x < \varepsilon.
   \]

The next Proposition illustrates the difference between algebra and analysis. In algebra we show equality directly. In analysis we sometimes show two numbers are the same by showing that their difference is arbitrarily small!

\textbf{Proposition 7} \textit{If} \( x, y \) \textit{are real numbers such that}

\[
|x - y| < 1/n \text{ for every } n \in \mathbb{N},
\]

\textit{then} \( x = y \).

\textbf{Proof:} We prove this by contradiction. Suppose the conclusion \( x = y \) is false.
Then since \( x \neq y \), it follows that \( x - y \neq 0 \).
Therefore \(|x - y| > 0\).
Now by Property 3, \(|x - y| > 1/n\) for some \( n \in \mathbb{N} \).
This contradicts the statement \(|x - y| < 1/n\) for every \( n \in \mathbb{N} \).
q.e.d.

\textbf{Lemma 6} \textit{For any real numbers} \( x \) \textit{and} \( y \):

1. \(|xy| = |x||y|\).
2. \(x^2 \geq 0\) and so \(|x^2| = x^2\).
3. \(|x + y| \leq |x| + |y| \) (The \textit{triangle inequality}).
4. $||x| - |y|| \leq |x - y|$.

**Proof:** We prove each statement separately.

1. $|xy|$ and $|x||y|$ are both non-negative numbers equal either to $xy$ or to $-xy$.
   Therefore in this case they are equal.

2. If $x \geq 0$ then this follows because the product of positives is positive.
   If $x < 0$ then $x = -|x|$ and $|x| > 0$. Thus $x^2 = |x|^2 > 0$.

3. Note that
   
   $$|x+y|^2 = (x+y)^2 = x^2+2xy+y^2 = |x|^2+2xy+|y|^2 \leq |x|^2+2|x||y|+|y|^2 = (|x|+|y|)^2.$$

   Therefore $|x+y|^2 \leq (|x|+|y|)^2$. It follows that

   $$0 \leq ((|x|+|y|)^2 - |x+y|^2) = \left[|x|+|y| - |x+y|\right]\left[\left(|x|+|y|\right) + |x+y|\right].$$

   Since the second factor on the right is not negative it follows that

   $$|x| + |y| - |x+y| \geq 0.$$

4. By the triangle inequality,

   $$|x-y| + |y| \geq |x-y+y| = |x|,$$

   and

   $$|x-y| + |y-x| \geq |x-y-x| = |y|.$$

   Therefore

   $$|x-y| \geq |x| - |y| \quad \text{and} \quad |x-y| \geq |y| - |y-x| = |y| - |x|$$

   Therefore

   $$|x-y| \geq ||x| - |y||.$$

q.e.d.

The next Proposition is **fundamental** for this course.

**Proposition 8** Suppose $a \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ is positive. Then for $x \in \mathbb{R}$

$$|x-a| < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon.$$
Proof: The statement $|x - a| < \varepsilon$ is the same as

$$x - a < \varepsilon \text{ and } a - x < \varepsilon.$$  

But

$$x - a < \varepsilon \iff x < a + \varepsilon \text{ and } a - x < \varepsilon \iff x > a - \varepsilon.$$  

Thus

$$|x - a| < \varepsilon \iff a - \varepsilon < x < a + \varepsilon.$$  

q.e.d.

Exercise 11 Absolute value

1. If $x$ is a positive real number show that for some $\varepsilon > 0,$

$$y \in \mathbb{R} \text{ and } |x - y| < \varepsilon \Rightarrow y > 0.$$  

2. If $x, z \in \mathbb{R}$ show that for each $\varepsilon > 0$ there is some $\delta > 0$ such that if $y \in \mathbb{R}$ satisfies $|y - x| < \delta$ then $|zy - zx| < \varepsilon.$ (Hint: Use $y^2 - x^2 = (y-x)(y+x).$)  

3. If $x \in \mathbb{R}$ show that for each $\varepsilon > 0$ there is some $\delta > 0$ such that if $y \in \mathbb{R}$ satisfies $|y - x| < \delta$ then $|y^2 - x^2| < \varepsilon.$ (Hint: Use $y^2 - x^2 = (y-x)(y+x).$)  

4. If $x \in \mathbb{R}$ and $x \neq 0$ show that for each $\varepsilon > 0$ there is some $\delta > 0$ such that if $y \in \mathbb{R}$ satisfies $|y - x| < \delta$ then $y \neq 0$ and $|1/y - 1/x| < \varepsilon.$

4.4 Bounds

Definition 20 Bounds

1. A nonvoid subset $S \subset \mathbb{R}$ is bounded above if for some $b \in \mathbb{R},$

$$x \leq b \text{ for all } x \in S.$$  

In this case we write

$$S \leq b \text{ or } b \geq S,$$

and say $b$ is an upper bound for $S.$  

2. A nonvoid subset $S \subset \mathbb{R}$ is bounded below if for some $a \in \mathbb{R},$

$$a \leq x \text{ for all } x \in S.$$  

In this case we write

$$a \leq S \text{ or } S \geq a,$$

and say $a$ is a lower bound for $S.$
3. A subset $S \subseteq \mathbb{R}$ is **bounded** if it is both bounded above and bounded below.

**Lemma 7** Suppose $S$ and $T$ are non-void subsets of $\mathbb{R}$. Then every element of $S$ is a lower bound for $T$ if and only if every element of $T$ is an upper bound for $S$.

In this case we write

$$S \leq T \quad \text{or} \quad T \geq S.$$  

**Proof:** The statement that every element of $S$ is a lower bound for $T$ is true if and only if $x \leq T$ for every $x \in S$.

This is equivalent to the statement:

$$x \leq y \quad \text{for all} \quad x \in S \quad \text{and all} \quad y \in T.$$

But this is equivalent to saying that every $y \in T$ is an upper bound for $S$.

q.e.d.

**Example 22** **Bounds**

1. Any real number $x \leq 1$ is a lower bound for $\mathbb{N}$.

2. $\mathbb{N}$ is not bounded above.

   In fact if $x$ is any real number then by Property 3.5 there is some natural number $m$ such that $x < m$.

   Thus $x$ is not an upper bound for $\mathbb{N}$, and so $\mathbb{N}$ does not have an upper bound.

3. $S = \{1/n \mid n \in \mathbb{N}\}$ is bounded above by any number $\geq 1$ and bounded below by any number $\leq 0$.

4. The rationals are not bounded below or above.

5. The set $S$ of real numbers whose squares are less than 2 is bounded above by 2 and below by $-2$.

   In fact, if $x > 2$ then $x^2 = xx > 2x > 4$ and so $x \notin S$.

   Also, if $x < -2$ then $-x > 2$ and $x^2 = xx = (-x)(-x) > 4$ and again, $x \notin S$.

   Thus

   $$-2 \leq S \leq 2.$$  

**Exercise 12** **Bounds**

1. Show that if $x$ is a positive real number then $x + 1$ is not a lower bound for $\mathbb{N}$.

2. Show that if $b$ is an upper bound for a non-void set $S \subseteq \mathbb{R}$ then every $c > b$ is also an upper bound for $S$. 

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3. Give an example of a non-void bounded set $S \subset \mathbb{R}$ which contains an upper bound but does not contain a lower bound.

4. Show that a non-void set $S \subset \mathbb{R}$ cannot contain two different upper bounds.

5. Show that the set of real numbers of the form $x/y$ with $|x| > |y| > 0$ is not bounded above or below.

6. Show that the set of real numbers of the form $x/y$ with $|y| > |x| > 0$ is bounded. Find explicit upper and lower bounds and prove your answers are correct.

7. Suppose $S \subset T \subset \mathbb{R}$ are non-void sets. If $S \leq T$ show that $S$ has exactly one element.

8. More generally, suppose $S \subset \mathbb{R}$ and $T \subset \mathbb{R}$ are non-void sets. If $S \leq T$ show that $S \cap T$ can have at most one element. If $S \cap T$ is non void show that its unique element is an upper bound for $S$ and a lower bound for $T$.

### 4.5 Least Upper and Greatest Lower Bounds

In everything we have done so far there is nothing to suggest that there are real numbers which are not rationals!

All we have assumed about the real numbers at this point is listed in Properties One, Two, and Three at the start of Section 4.2.

We correct this with one final property, still as an axiom. This is the fundamental property of the real numbers and is the key fact which makes analysis possible.

**Property Four:** Let $S$ be a non empty subset of $\mathbb{R}$ which is bounded above. Then $S$ has an upper bound, $b$ with the following property:

$$y \geq b$$

**Lemma 8** The upper bound $b$ in Property Four is unique.

**Proof:** Suppose $c$ satisfies the same condition as $b$. Then $c$ is an upper bound for $S$. Therefore $c \geq b$. But $b$ is an upper bound for $S$. Therefore $b \geq c$. Therefore $b = c$.

q.e.d.

**Definition 21** The real number $b$ in Property Four is called the least upper bound for $S$ and is denoted by lub$(S)$ or by sup$(S)$. 
Important Remark: If \( S \) is bounded above, sometimes \( \text{lub}(S) \) is in \( S \) and sometimes it is not! For example if

\[
S = \{ x \in \mathbb{R} \mid x \leq 1 \}
\]

or if

\[
S = \{ x \in \mathbb{R} \mid x < 1 \},
\]
then in both cases \( \text{lub}(S) = 1 \). However, in the first case, \( 1 \in S \) and in the second case \( 1 \notin S \).

Analogous to least upper bounds, a non empty set which is bounded below has a greatest lower bound:

**Proposition 9** Suppose \( S \) is a non empty subset of \( \mathbb{R} \) which is bounded below. Then there is a unique lower bound, \( a \) with the following property:

\[ \text{If } y \text{ is any lower bound for } S \text{ then } y \leq a. \]

**Definition 22** The real number \( a \) in Proposition 9 is called the greatest lower bound for \( S \) and is denoted by \( \text{glb}(S) \) or by \( \text{inf}(S) \).

**Proof of Proposition 9:** Let \( -S = \{ -x \mid x \in S \} \).

If \( y \) is a lower bound for \( S \) then \( y \leq S \).

Since multiplication by \(-1\) reverses inequalities, \(-y \geq -S\).

Therefore \(-S\) is bounded above, and so by Property Four, it has a least upper bound, \( b \).

Set \( a = -b \).

Since \( b \geq -S \), it follows that \( a = -b \leq S \).

Now let \( y \) be any lower bound for \( S \). Then \( y \leq S \) and so

\[ -y \geq -S. \]

Thus \(-y\) is an upper bound for \(-S\) and so \(-y \geq b\).

Therefore \( y \leq -b = a \) and so \( a = \text{glb}(S) \). Its uniqueness is proved in the exact same way as the uniqueness of \( \text{lub}\)'s.

\[ \text{q.e.d.} \]

We recap what it means for \( b \) to be the least upper bound of a non-void set \( S \). It means exactly this:

\[ S \leq b \]

and

If \( u < b \) then \( u \) is not an upper bound for \( S \)

A useful way of expressing this is given in the next Proposition:
Proposition 10 Suppose $S$ is a non-void subset of $\mathbb{R}$. Then

1. $b$ is the least upper bound of $S$ if and only if
   
   (a) $S \leq b$, and
   
   (b) for all $\varepsilon > 0$ there is an $x \in S$ such that $|b - x| < \varepsilon$.

2. $a$ is the greatest lower bound of $S$ if and only if

   (a) $S \geq a$, and

   (b) for all $\varepsilon > 0$ there is an $x \in S$ such that $|a - x| < \varepsilon$.

Proof: For the first assertion, suppose first that (a) and (b) are satisfied. Then $b$ is an upper bound. To show that $b$ is the lub we need to show that if $u < b$ then $u$ is not an upper bound for $S$.

But if $u < b$, then $b - u > 0$. Therefore by hypothesis (b), for some $x \in S$, $|b - x| < b - u$. Now because $b$ is an upper bound for $S$ we also have $b - x > 0$ and so

$$ b - x = |b - x| < b - u. $$

Thus $x > u$ and $u$ is not an upper bound for $S$.

Conversely, suppose $b$ is the least upper bound for $S$. Then $b \geq S$.

Moreover, if $\varepsilon > 0$, then $b - \varepsilon < b$ and so $b - \varepsilon$ is not an upper bound for $S$. Therefore, for some $x \in S$

$$ b - \varepsilon < x. $$

It follows that $|b - x| = b - x < \varepsilon$ and so (b) is true.

The second assertion is proved in the same way.

q.e.d.

Example 23

1. $\text{glb}(\mathbb{N}) = 1$.
   
   In fact, since $1 \in \mathbb{N}$, any lower bound $x$ for $\mathbb{N}$ satisfies $x \leq 1$
   
   But $1 \leq n$ for all $n \in \mathbb{N}$. Thus $1$ is a lower bound for $\mathbb{N}$ and so it is the
   greatest lower bound: $\text{glb}(\mathbb{N}) = 1$.

2. Let $S = \{(x + 1)/x \mid x \in \mathbb{R} \text{ and } x \geq 1\}$. Then $1 = \text{glb}(S)$.
   
   In fact for $x \geq 1$

   $$ 1 \leq 1 + 1/x. $$

   Moreover, if $y > 1$ then by Property 3 there is some $n \in \mathbb{N}$ with $1/n < y - 1$. Thus

   $$ 1 + 1/n < y $$

   and so $y$ is not a lower bound for $S$. Thus $1$ is the greatest lower bound.
Exercise 13 \( \text{glb} \) and \( \text{lub} \)

1. If \( b \in S \) is an upper bound for a non-void set \( S \subset \mathbb{R} \) show that it is the least upper bound.

2. Show that 0 is the greatest lower bound for \( \{1/n \mid n \in \mathbb{N}\} \).

3. Let \( a \in \mathbb{R} \) and let \( S = \{x \in \mathbb{R} \mid |x - a| < 1/2\} \). Show that \( S \) is bounded. Then find \( \text{lub}(S) \) and \( \text{glb}(S) \) and prove your answers are correct.

4. Let \( S \) be a a non-void subset of \( \mathbb{R} \) which is bounded above. Show that the following conditions are equivalent on an upper bound, \( b \), for \( S \):
   
   (a) \( b = \text{lub}(S) \).
   
   (b) For each \( \varepsilon > 0 \) there is an element \( x \in S \) such that \( b - \varepsilon < x \leq b \).
   
   (c) For each \( \varepsilon > 0 \) there is an element \( x \in S \) such that \( 0 \leq b - x < \varepsilon \).

5. Suppose \( S \subset T \) are subsets of \( \mathbb{R} \):
   
   (a) If \( T \subset \mathbb{R} \) is a non-void set bounded below show that \( S \) is bounded below and that \( \text{glb}(T) \leq \text{glb}(S) \);
   
   (b) If \( T \) is bounded above show that \( S \) is bounded above and that \( \text{lub}(T) \geq \text{lub}(S) \).

4.6 Powers

The integral powers of a real number \( x \neq 0 \) are defined by induction as follows:

**Definition 23** Let \( x \) be a non-zero real number. If \( n \in \mathbb{N} \) then

\[
x^1 = x \quad \text{and} \quad x^{n+1} = x \cdot x^n.
\]

Then we set \( x^0 = 1 \) and \( x^{-n} = 1/x^n \).

Our objective here is to extend the definition to rational powers of a positive real number, \( x \). To do this we first need to prove:

**Theorem 5** Let \( \mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\} \) be the set of positive real numbers. Then for any natural number \( n \in \mathbb{N} \):

1. For \( x, y \in \mathbb{R}_+ \), \( x < y \iff x^n < y^n \).

2. The map \( \mathbb{R}_+ \to \mathbb{R}_+, \ x \mapsto x^n \) is a 1-1 correspondence.

**Proof:**
1. The Difference theorem gives

\[ y^n - x^n = (y - x) \sum_{i=0}^{n-1} y^i x^{n-1-i}. \]

Since both \( x \) and \( y \) are positive, \( \sum_{i=0}^{n-1} y^i x^{n-1-i} \) is positive. Therefore

\[ y^n - x^n > 0 \iff y - x > 0. \]

2. First observe that because of the assertion above, the map is 1-1.

Now we prove that our map is onto.

We need to show that if \( y \) is a positive real number and \( n \in \mathbb{N} \) then there is a positive real number \( x \) such that \( x^n = y \).

For this define a set \( T \) by

\[ T = \{ x \in \mathbb{R}_+ \mid x^n > y \}. \]

Observe that \( T \neq \emptyset \). In fact, since \( y > 0 \), it follows that \( y + 1 > 1 \) and so \( (y + 1)^n \geq y + 1 > y \). Thus \( y + 1 \in T \).

On the other hand, \( 0 < T \) and so \( T \) is bounded below. Therefore \( T \) has a greatest lower bound, \( \text{glb}(T) \). Write \( x = \text{glb}(T) \). We show that

\[ x^n = y. \]

By Property 3, exactly one of the following three possibilities must hold:

\[ x^n < y, \quad x^n > y, \quad \text{or} \quad x^n = y. \]

To prove our assertion we show by contradiction that the two inequalities above are false, and so it must be true that \( x^n = y \). The strategy is as follows:

(a) We show that if \( x^n < y \) then \( x \) is not the greatest lower bound of \( T \), which is a contradiction.

To do this, we will show that if \( x^n < y \) then there is a positive real number \( c \) such that \( x^n < c^n < y \).

Indeed, if such a \( c \) exists then for \( z \in T \),

\[ c^n < y < z^n, \]

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and so by the first assertion of the Proposition,
\[ c < z \quad \text{for all} \quad z \in T. \]

Thus \( c \) is a lower bound for \( T \). But since \( x^n < c^n \) we have \( x < c \) and so \( x \) would not be the greatest lower bound.

(b) We show that if \( x^n > y \) then \( x \) is not a lower bound of \( T \), which is also a contradiction.

To do this, we will show that if \( x^n > y \) then there is a positive real number \( b \) such that \( y < b^n < x^n \).

But if such a \( b \) exists then by the definition of \( T \), \( b \in T \). Now by the first assertion of the Proposition,
\[ b < x, \]
and so \( x \) is not a lower bound for \( T \).

**In summary**, if we can construct \( c \) and \( b \) as above then the possibilities \( x^n < y, \ x^n > y \), are excluded and so it follows that \( x^n = y \).

It remains to construct \( c \) and \( b \).

**To construct** \( c \) so that \( x^n < c^n < y \) recall that we are assuming \( x^n < y \).

Thus we may choose \( k \in \mathbb{N} \) so that \( 1/k < x \) and so that
\[ 1/k < \frac{y - x^n}{2^n x^{n-1}}. \]

Set \( c = x + 1/k \). Then \( c > x \) and so
\[ c^n > x^n \quad \text{and} \quad (1/k)x^{n-1} 2^n < y - x^n. \]

Now, by the Binomial theorem,
\[ c^n - x^n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} x^i (1/k)^{n-i} - x^n = 1/k \sum_{i=0}^{n-1} \frac{n!}{i!(n-i)!} x^i (1/k)^{n-i-1}. \]

Since \( 0 < 1/k < x \) it follows that \( x^i (1/k)^{n-i-1} < x^{n-1} \).

Therefore, again by the Binomial Theorem,
\[ c^n - x^n < 1/k \sum_{i=0}^{n-1} \frac{n!}{i!(n-i)!} x^{n-1} < (1/k)x^{n-1} \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} = (1/k)x^{n-1}(1+1)^n. \]
Thus
\[ c^n - x^n < (1/k)x^{n-1}\frac{1}{2^n} < y - x^n, \]
and so \( c^n < y \). This completes the construction of \( c \).

**To construct** \( b \) recall that we are assuming \( x^n > y \).

Thus we may choose \( k \in \mathbb{N} \) so that \( 0 < 1/k < x \) and
\[ 1/k < \frac{x^n - y}{nx^{n-1}}. \]

Set \( b = x - 1/k \). Then \( 0 < b < x \) and so
\[ x - b = 1/k \quad \text{and} \quad b^n < x^n. \]

Thus by the Difference theorem, since \( x - b = 1/k \),
\[ x^n - b^n = (x - b) \sum_{i=0}^{n-1} x^i b^{n-1-i} \leq (x - b)nx^{n-1} = \frac{nx^{n-1}}{k} < x^n - y. \]
Therefore \( y < b^n \).

This completes the construction of \( b \) and the proof of the Theorem.

q.e.d.

**Definition 24** If \( y \) is a positive real number and \( n \) is a natural number then the unique positive number \( x \) such that \( x^n = y \) is called the \textbf{nth root of} \( y \) and is denoted by \( x = y^{1/n} \).

This Theorem allows us to construct our first real number that is not rational. It shows that there is a unique positive real number \( x \) such that \( x^2 = 2 \). Thus now we know that \( \sqrt{2} \) exists. On the other hand, according to Exercise 2.4, \( x \) is not a rational number.

We can now define any rational power of a positive real number. For this we need the following:

**Lemma 9** Suppose \( n, q \in \mathbb{N} \) and \( m, p \in \mathbb{Z} \). If \( x \) is a positive real number and \( m/n = p/q \) then
\[ (x^{1/n})^m = (x^{1/q})^p. \]
Proof: First note that

\[(x^{1/n})^{mq} = (x^{1/n})^{mnq} = ((x^{1/n})^n)^{mq} = x^{mq}.\]

In the same way,

\[(x^{1/q})^{np} = (x^{1/q})^{qpn} = ((x^{1/q})^q)^{pn} = x^{pn}.\]

Since \(m/n = p/q\) it follows that \(mq = pn\). Therefore

\[((x^{1/n})^m)^{nq} = ((x^{1/q})^p)^{nq}.\]

By the Theorem above, raising to the \((nq)\)th power is a 1-1 map. Therefore

\[(x^{1/n})^m = (x^{1/q})^p,\]

and the Lemma is proved.

q.e.d.

Now let \(a\) be any rational number. We can express \(a\) in the form \(a = m/n\) with \(m \in \mathbb{Z}\) and \(n \in \mathbb{N}\). Lemma 9 shows that if \(x\) is a positive real number then \((x^{1/n})^m\) does not depend on the choice of \(m\) and \(n\). Thus we may make the

Definition 25 Let \(x\) be any positive real number and let \(a\) be any rational number. Then

\[x^a = (x^{1/n})^m,\]

where \(a = m/n\) and \(m \in \mathbb{Z}\) and \(n \in \mathbb{N}\).

Exercise 14 Powers

1. Show that if \(a, b\) are rational numbers and \(x\) is a positive real number then \(x^a x^b = x^{a+b}\).

2. Show that if \(a, b\) are rational numbers and \(x\) is a positive real number then \((x^a)^b = x^{ab}\).

3. Show that if \(x \in \mathbb{R}\) satisfies \(x > 1\) and if \(c \in \mathbb{Q}\) is positive, then \(x^c > 1\).

4. Show that if \(x > 1\) is a real number and if \(a < b\) are positive rational numbers then \(0 < x^a < x^b\).

5. Show that if \(a \in \mathbb{Q}\) is positive and if \(0 < x < y\) then \(x^a < y^a\).

6. If \(k \in \mathbb{N}\) and \(x > 0\) is a real number, show by induction on \(k\) that

\[x^k \leq x \text{ if } x < 1 \text{ and } x^k \geq x \text{ if } x > 1.\]

7. If \(x, \varepsilon \in \mathbb{R}\) are positive and if \(x < 1\) show that for some \(N \in \mathbb{N}\),

\[x^a < \varepsilon \text{ if } a \in \mathbb{Q} \text{ and } a \geq N.\]
8. Let \( p < q \in \mathbb{N} \) and \( x > 0 \) be a real number. For which \( x \) is \( x^{p/q} > x \) and for which \( x \) is \( x^{p/q} < x \)? Prove your answer is correct.

9. Let \( x \in \mathbb{R} \) and let \( S \) be the set of those rationals of the form \( \frac{p}{10^k} \) satisfying the three conditions: (i) \( p \in \mathbb{Z} \), (ii) \( k \in \mathbb{N} \), and (iii) \( \frac{u}{10^k} < x \). Show that \( x = \text{lub}(S) \). Hint: Follow the following steps:

   (a) Let \( \varepsilon > 0 \) be an arbitrary positive number. Use Proposition 5 to conclude that there are rational numbers \( a, b \) such that \( x - \varepsilon < a < b < x \).

   (b) Show that for some \( k \in \mathbb{N} \), \( 10^k b - 10^k a > 2 \). Explain why this implies that some integer \( p \) satisfies \( 10^k a < p < 10^k b \).

   (c) Conclude that \( \frac{p}{10^k} \in S \) and that \( x - \varepsilon < \frac{p}{10^k} < x \).

4.7 Constructing the real numbers

Call a set \( S \) of rationals that is bounded above \textit{full} if

\[
y \in S \text{ and } z < y \implies z \in S.
\]

Then simply \textbf{define} the real numbers to be the subsets of the rationals that are bounded above and full, and identify each rational \( \frac{p}{q} \) with the set of all rationals \( \leq \frac{p}{q} \). Intuitively we have just "filled in the holes" among the rationals.

If \( S \) and \( T \) are sets of rationals that are bounded above and full, then we think of these sets as real numbers \( x \) and \( y \) and define \( x < y \) if \( S \subseteq T \), and \( x > y \) if \( S \supseteq T \). Then set \( x + y \) to be the set

\[
S + T = \{ p/q + m/n \mid p/q \in S \text{ and } m/n \in T \},
\]

and define the other algebraic operations in a similar way.

Of course, we then need to prove that all these definitions are well-defined, that all algebraic rules are still true, that the properties above for the order hold, and that the operations and order for the rationals remain as before. And then, of course, we need to prove Properties One through Four. All this is long and boring, so instead we will simply take for granted the existence of the real numbers satisfying those properties.
Chapter 5

Infinite Sequences

5.1 Convergent sequences

Definition 26 An infinite sequence \((x_i)_{i \geq k}\) of elements in a set \(S\) is a list of elements \(x_i \in S\), with \(i \in \mathbb{Z}\) and \(i \geq k\).

We say the sequence begins with \(x_k\). For any \(p \geq k\), the infinite sequence \((x_i)_{i \geq p}\) is called the \(p\)th tail of the original sequence.

Remark: The set of elements appearing in an infinite sequence may be finite, as illustrated by the infinite sequence \(1, 0, 1, 0, 1, 0, \ldots\) in which only two integers, 0 and 1 appear.

In the world of applications we never need to know the exact value of a real number - we only need to know it within a given tolerance. Specs for any engineering design always specify heights, lengths, weights etc. up to so many fractions of an inch or so many millimeters, or so many fractions of a gram.

Infinite sequences \((x_i)\) of real numbers are therefore a fundamental tool for applications because they can be used to approximate a real number \(x\) to within a given tolerance. Intuitively that means that the error, \(|x_i - x|\), gets arbitrarily small as \(i\) gets larger. In other words, if we want to approximate \(x\) within a given tolerance we may simply use one of the \(x_i\) as long as \(i\) is sufficiently large.

Infinite sequences are also a fundamental tool in analysis, and for this we need as always to formalize the intuitive idea above:

Definition 27 An infinite sequence \((x_i)_{i \geq k}\) of real numbers converges to \(x \in \mathbb{R}\) if for each \(\varepsilon > 0\) there is some integer \(N \geq k\) (usually depending on \(\varepsilon\)) such that

\[ i \geq N \Rightarrow |x_i - x| < \varepsilon. \]

Lemma 10 If an infinite sequence \((x_i)\) of real numbers converges, it converges to a unique real number \(x\).
**Proof:** Suppose the sequence \((x_i)\) converges to both \(x\) and \(y\).
Then for each \(\varepsilon > 0\) there is some \(N\) such that for \(i \geq N\) both \(|x_i - x| < \varepsilon/2\)
and \(|x_i - y| < \varepsilon/2\).

It follows that
\[
|x - y| = |x - x_i + x_i - y| \leq |x_i - x| + |x_i - y| < \varepsilon.
\]
Since this is true for each \(\varepsilon\), \(x = y\).

q.e.d.

**Definition 28** If \((x_i)\) is a convergent sequence, the unique \(x\) to which it converges is called the **limit** of the sequence and is denoted by \(\lim_{i}(x_i)\).

**Example 24** The infinite sequence \(1, 2, 3, \ldots\) of natural numbers does not converge.

**Proof:** This is proved by contradiction. Assume the sequence converges to some \(x\). Then for some \(N\),
\[
n \geq N \Rightarrow |x_n - x| < 1.
\]
Thus for \(n \geq N,\)
\[
|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|.
\]
But by Property Three, for some \(k,\)
\[
1 + |x| < k < k + N = x_{k+N}.
\]
This is the desired contradiction. q.e.d.

**Example 25**

1. The infinite sequence \(0, 1, -1, 2, -2, 3, -3, \ldots\) lists all the integers. This sequence does not converge.

2. The infinite sequence \((x_n)_{n \geq 1}\) defined by \(x_n = 1/n\). This sequence converges to 0.

3. The infinite sequence \((x_n)_{n \geq 1}\) defined inductively by \(x_1 = 1\), and
\[
x_n = \sum_{i=1}^{n-1} 2^x_i.
\]
This sequence does not converge.

4. The infinite sequence \((x_n)_{n \geq 1}\) defined by: \(x_n\) is the \(n\)th largest prime number. This sequence does not converge.

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5. The infinite sequence \((x_n)_{n \geq 1}\) defined by: \(x_n = 1+(-1/2)^n\). This sequence converges to 1.

Lemma 11 If \((x_i)\) and \((y_i)\) are infinite sequences of real numbers converging respectively to \(x\) and \(y\), then \((x_iy_i)\) converges to \(xy\).

Proof: If \(\varepsilon > 0\), choose \(N \in \mathbb{N}\) so that if \(n \geq N\) then
\[
|x_n - x| < \frac{\varepsilon}{2(|y|+1)}, \quad |y_n - y| < \frac{\varepsilon}{2(|x|+1)},
\]
and
\[
|y_n - y| < 1.
\]
Then for \(n \geq N\),
\[
|y_n| = |y_n - y + y| \leq |y_n - y| + |y| < |y| + 1.
\]
Therefore for \(n \geq N\),
\[
|x_n y_n - xy| = |x_n y_n - x y_n + x y_n - xy| \leq |x_n - x||y_n| + |x||y - y_n| < \frac{\varepsilon}{2(|y|+1)}(|y|+1) + |x| \frac{\varepsilon}{2(|x|+1)} \leq \varepsilon.
\]
q.e.d

Exercise 15 Convergence
As always, provide complete proofs for your answers.

1. In each of the examples in the second Example above supply proofs for the statements about convergence.

2. Suppose \((x_n)_{n \geq 1}\) is an infinite sequence of real numbers converging to \(x\). Define the sequence \((y_n)_{n \geq 1}\) by \(y_n = x_{2n}\). Is this sequence convergent, and if so, what is its limit? Prove your answer is correct.

3. Suppose \((x_n)_{n \geq 1}\) is an infinite sequence of real numbers. Suppose the sequence \(y_n = x_{3n}\) converges to \(y\) and that the sequence \(z_n = x_{3n+2}\) converges to \(z\). If \(y \neq z\) show that the original sequence \((x_n)\) is not convergent.

4. Show that if \(x\) is any real number then there is an infinite sequence of rational numbers converging to \(x\).

5. Suppose \((x_n)_{n \geq 1}\) is an infinite sequence of real numbers converging to \(x\). Define a sequence \((y_n)_{n \geq 1}\) by \(y_n = x_{n+1} - x_n\). Prove that the sequence \((y_n)_{n \geq 1}\) is convergent, and find its limit. Prove your answer is correct.

6. Suppose \((x_i)\) and \((y_i)\) are infinite sequences of real numbers converging respectively to \(x\) and \(y\).

(a) Show that \((x_i + y_i)\) converges to \(x + y\).

(b) If \(y \neq 0\) show that for some \(n \in \mathbb{N}\), \(y_i \neq 0\) for \(i \geq n\). In this case prove that the infinite sequence \((1/y_i)_{i \geq n}\) converges to \(1/y\).
(a) If \((x_i)\) is a convergent sequence and if each \(x_i \leq b\), show that \(\lim_i(x_i) \leq b\).

(b) If \((x_i)\) is a convergent sequence and if each \(x_i \geq a\), show that \(\lim_i(x_i) \geq a\).

7. If \((x_i)\) is an infinite sequence converging to \(x\), show that the sequence \(|x_i|\) is convergent, and find its limit.

## 5.2 Bounded sequences

**Definition 29**

1. An infinite sequence, \((x_i)_{i \geq k}\), of real numbers is **bounded above** if for some real number \(b\),
   \[ x_i \leq b \quad \text{for all} \quad i \geq k. \]

2. An infinite sequence, \((x_i)_{i \geq k}\), of real numbers is **bounded below** if for some real number \(a\),
   \[ a \leq x_i \quad \text{for all} \quad i \geq k. \]

3. An infinite sequence is **bounded** if it is both bounded above and bounded below.

**Definition 30**

1. An infinite sequence, \((x_i)_{i \geq k}\), of real numbers is **increasing** if each \(x_i \leq x_{i+1}\).

2. An infinite sequence, \((x_i)_{i \geq k}\), of real numbers is **decreasing** if each \(x_i \geq x_{i+1}\).

**Lemma 12**

1. An increasing sequence, \((x_i)_{i \geq k}\) that is bounded above converges, and
   \[ \lim_i(x_i) = \text{lub}\{x_i \mid i \geq k\}. \]

2. A decreasing sequence, \((y_i)_{i \geq k}\) that is bounded below converges, and
   \[ \lim_i(y_i) = \text{glb}\{y_i \mid i \geq k\}. \]

**Proof:**

1. Set \(S = \{x_i \mid i \geq k\}\).
   Then \(S\) is bounded above.
   Fix any \(\varepsilon > 0\).
   By the definition of \(\text{lub}(S)\), \(\text{lub}(S) - \varepsilon\) is not an upper bound for \(S\).
   Therefore there is some \(x_p\) such that
   \[ x_p > \text{lub}(S) - \varepsilon. \]
Fix such a $p$.

Since the sequence is increasing, for any $i \geq p$,

$$x_i > \text{lub}(S) - \varepsilon.$$  

On the other hand, since each $x_i \in S$, for $i \geq p$

$$x_i \leq \text{lub}(S).$$

Therefore, for $i \geq p$,

$$0 \leq \text{lub}(S) - x_i < \varepsilon.$$  

In particular,

$$|\text{lub}(S) - x_i| < \varepsilon, \quad i \geq p.$$  

Since $\varepsilon$ was any positive real number, this is precisely the statement that the sequence converges to $\text{lub}(S)$.

2. Set $T = \{y_i \mid i \geq k\}$.

Then $T$ is bounded below.

Set $-T = \{-y_i \mid i \geq k\}$.

Since $T$ is bounded below and multiplication by $-1$ reverses inequalities, $-T$ is bounded above and

$$\text{lub}(-T) = -\text{glb}(T).$$

Moreover, the sequence $(y_i)_{i \geq k}$ is increasing.

Thus by the first part of the Lemma, if $\varepsilon > 0$ there is a $p \in \mathbb{N}$ such that

$$|\text{lub}(-T) - (\text{glb}(T))| < \varepsilon, \quad i \geq p.$$  

Since $\text{lub}(-T) = -\text{glb}(T)$ it follows that $\text{glb}(T) = -\text{lub}(-T)$. Therefore, for $i \geq p$,

$$|y_i - \text{glb}(T)| < \varepsilon, \quad i \geq p.$$  

Thus the sequence $(y_i)_{i \geq k}$ converges to $\text{glb}(T)$.

q.e.d.

Exercise 16

1. Show that convergent sequences are bounded. Hint: If $x_n$ converges to $x$ then for some $N$, $x - 1 < x_n < x + 1$ if $n \geq N$. Use this to prove that the sequence $(x_n)_{n \geq N}$ is bounded. Then show that the entire sequence is bounded.
2. Which of the following infinite sequences are bounded, which are increasing, which are decreasing, and which converge to a limit? If the sequence is convergent, find the limit. As always prove your answers.

(a) \(1, -1, 1, -1, 1, -1, \ldots\)
(b) \((x_q)_{q \geq 1} = x^{1/q}\) where \(0 < x < 1\) is fixed.
(c) \((x_q)_{q \geq 1} = x^{1/q}\) where \(x = 1\).
(d) \((x_q)_{q \geq 1} = x^{1/q}\) where \(x > 1\) is fixed.

3. Suppose \(a \in \mathbb{Q}\) is positive. If \((x_i)\) is a sequence of positive real numbers converging to \(x > 0\), show that the sequence \((x_i^a)\) converges and find its limit, proving your answer. Is the converse true? Hint: use the following steps:

(a) Show by induction that for \(n \in \mathbb{N}\), \((x_i^n)\) converges and find its limit.
(b) Show that for \(n \in \mathbb{N}\), \((x_i^{2/n})\) converges and find its limit.
(c) Complete the proof when the exponent is a positive rational.

4. Suppose \(x\) is a real number and that \(0 < x < 1\).

(a) Show that the set \(S = \{x^n \mid n \in \mathbb{N}\}\) is bounded below.
(b) Show that \(\text{glb}(S) = \text{glb}\{x^n \mid n \geq 2\}\).
(c) Show that \(\text{glb}(S) = x \cdot \text{glb}(S)\), and conclude that \(\text{glb}(S) = 0\).
(d) Prove that the sequence \((x^n)_{n \geq 1}\) converges and find its limit. Prove that this limit is correct.

Proposition 11 (Cauchy criterion) A sequence \((x_n)\) of real numbers converges if and only if for each \(\varepsilon > 0\) there is some integer \(N\) such that \(|x_n - x_m| < \varepsilon\) for all \(n, m \geq N\).

Proof: Suppose first that the sequence converges to \(x\).

Then for any \(\varepsilon > 0\), there is some \(N\), such that

\[|x_i - x| < \varepsilon/2\text{ if }i \geq N.\]

Thus if \(n, m \geq N\),

\[|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \varepsilon.\]

Conversely, suppose that for each \(\varepsilon > 0\) there is some integer \(N\) such that

\[|x_n - x_m| < \varepsilon\text{ if }n, m \geq N.\]

Define sets \(S_n\) by setting

\[S_n = \{x_i \mid i \geq n\}.\]
We first show that $S_1$ is bounded. By hypothesis, for some $N$, if $n \geq N$, then

$$|x_n - x_N| < 1.$$  

Thus for $n \geq N$,

$$|x_n| \leq |x_n - x_N| + |x_N| < 1 + |x_N|.$$  

Since for $n < N, |x_n| \leq \max\{|x_1|, \ldots, |x_{N-1}|\}$, it follows that for all $n$,

$$|x_n| < 1 + |x_N| + \max\{|x_1|, \ldots, |x_{N-1}|\}.$$  

Thus $S_1$ is bounded.

Now, since each $S_n \subset S_1$, it follows that each $S_n$ is bounded. Thus we may set

$$a_n = \operatorname{glb}(S_n).$$

Since each $S_{n+1} \subset S_n$, it follows that $a_n$ is a lower bound for $S_{n+1}$. But $a_{n+1}$ is the greatest lower bound for $S_{n+1}$, and therefore

$$\cdots \leq a_n \leq a_{n+1} \leq \cdots$$

In other words, $(a_n)$ is an increasing sequence.

Moreover, since $a_n$ is a lower bound for $S_n$, for all $n$ we have

$$a_n \leq x_n < 1 + |x_N| + \max\{|x_1|, \ldots, |x_{N-1}|\}.$$  

Thus the sequence $a_n$ is bounded above. Apply Lemma 12 to conclude that the sequence $a_n$ is convergent to $a = \operatorname{lub}\{a_n\}$.

Finally, fix any $\varepsilon > 0$ and choose $N$ so that

$$|a_N - a| < \varepsilon/3 \quad \text{and} \quad |x_n - x_m| < \varepsilon/3 \quad \text{if} \quad n, m \geq N.$$  

Since $a_N$ is the glb for $S_N$ it follows that for some $m \geq N$ we have

$$|x_m - a_N| < \varepsilon/3$$

Therefore, for $n \geq N$,

$$|x_n - a| \leq |x_n - x_m| + |x_m - a_N| + |a_N - a| < \varepsilon.$$  

In other words, the sequence $x_n$ converges to $a$. \textbf{q.e.d.}

### 5.3 The Intersection Theorem

We now come to a result, which again uses Property Four about the reals. To state the Theorem we recall the

\textbf{Notation:} If $a \leq b$ are real numbers, then $[a, b]$ denotes the finite closed interval

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$  

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Theorem 6 (Intersection Theorem) Let

\[ S_m \supset S_{m+1} \supset S_{m+2} \supset \cdots \supset S_k \supset \cdots \]

be an infinite sequence of finite closed intervals:

\[ S_k = [a_k, b_k]. \]

Then

1. \((a_k)\) is an increasing sequence bounded above, and convergent to \(a = \text{ lub}\{a_k\}\);
2. \((b_k)\) is a decreasing sequence bounded below, and convergent to \(b = \text{ glb}\{b_k\}\);
3. \(b - a \geq 0\);
4. \((b_k - a_k)\) is a decreasing sequence of non-negative numbers convergent to \(b - a\);
5. \(\bigcap_k [a_k, b_k] = [a, b].\)

**Proof:** By definition, \(a_k \leq b_k\). Since \([a_k, b_k] \supset [a_{k+1}, b_{k+1}]\) it follows that \(a_k\) is a lower bound for \(S_{k+1}\) and \(b_k\) is an upper bound for \(S_{k+1}\) But by definition, \(a_{k+1}\) is the greatest lower bound for \(S_{k+1}\) and \(b_{k+1}\) is the least upper bound for \(S_{k+1}\). Therefore

\[ a_k \leq a_{k+1} \leq b_{k+1} \leq b_k. \]

Thus \((a_k)\) is an increasing sequence and \((b_k)\) is a decreasing sequence.

Moreover, since

\[ a_k \leq b_k \leq b_1, \]

the sequence \((a_k)\) is both increasing and bounded above. Therefore this sequence converges to \(a = \text{ lub}\{a_k\}\).

Similarly,

\[ a_1 \leq a_k \leq b_k. \]

Thus the sequence \((b_k)\) is decreasing and bounded below. It follows that the sequence \((b_k)\) converges to \(b = \text{ glb}\{b_k\}\).

In particular the sequence \((b_k - a_k)\) converges to \(b - a\), and since each \(b_k - a_k \geq 0\), it follows that \(b - a \geq 0\).

It remains to show that

\[ \bigcap_k [a_k, b_k] = [a, b].\]
But if \(x \in [a, b]\) then for all \(k\),

\[
    a_k \leq a \leq x \leq b \leq b_k,
\]

and so \(x \in [a_k, b_k]\). Thus \(x \in \bigcap_k [a_k, b_k]\); i.e.,

\[
    \bigcap_k [a_k, b_k] \supset [a, b].
\]

On the other hand, suppose \(x \in \bigcap_k [a_k, b_k]\). Then for all \(k\)

\[
    a_k \leq x \leq b_k.
\]

Thus \(x\) is an upper bound for the set \(\{a_k\}\) and a lower bound for the set \(\{b_k\}\).

It follows that

\[
    x \geq a \quad \text{and} \quad x \leq b.
\]

Thus \(x \in [a, b]\), and

\[
    \bigcap_k [a_k, b_k] \subset [a, b].
\]

This completes the proof of the Theorem.

q.e.d.

**Corollary** In the Theorem, suppose \(\lim_k (b_k - a_k) = 0\). Then

\[
    \bigcap_k [a_k, b_k]
\]

is a single point \(c\). If \((x_k)\) is an infinite sequence with \(x_k \in [a_k, b_k]\) then the sequence \((x_k)\) converges to \(c\).

**Proof:** It follows from the Theorem that \(b - a = 0\); i.e., \(b = a\).

Denote this point by \(c\).

Then \([a, b] = \{c\}\), and so

\[
    \bigcap_k [a_k, b_k] = \{c\}.
\]

Now, let \(\varepsilon > 0\) be any positive number.

Then choose \(N\) so that \(b_k - a_k < \varepsilon\) for \(k \geq N\).

Since \(c \in [a_k, b_k]\) and \(x_k \in [a_k, b_k]\) it follows that

\[
    k \geq N \implies |x_k - c| < \varepsilon.
\]

Thus the sequence \((x_k)\) converges to \(c\).

q.e.d.

**Exercise 17**
1. Construct a sequence of open intervals \( I(n) \) of length \( l(n) \) such that the sequence \((l(n))\) converges to zero, and such that each \( I(n) \) and \( I(n+1) \) have a point in common, but such that

\[
\bigcap_n I(n) = \emptyset.
\]

2. Let \( S_n \) be the set of rational numbers \( a \) satisfying \( \sqrt{2} - 1/n < a < \sqrt{2} + 1/n \). Find \( \bigcap_n S_n \).

3. Let \( S_n \) be the sets in the previous problem. If \( x_n \in S_n \), does the sequence \((x_n)\) converge? If so, find its limit.

4. Let \( S_n \) be the set of non-zero numbers in \([-1/n, 1/n]\). What is the intersection of these sets?

5. Let \( S_n \) be the open interval \((1, 1+1/n)\). What is the intersection of these sets?

5.4 Subsequences

**Definition 31** A subsequence of a sequence \((x_n)_{n \geq k}\) is a sequence of the form \((y_i = x_{n_i})_{i \geq r}\), in which \(n_r < n_{r+1} < n_{r+2} \ldots\) is an infinite increasing sequence of integers.

**Exercise 18** Subsequences

1. Construct a sequence which has two convergent subsequences converging to different limits.

2. Construct a sequence which for each \(n \in \mathbb{N}\) has a subsequence converging to \(n\).

3. Show that any subsequence of a convergent sequence converges to the same limit.

4. Show that a sequence which is not bounded above has an increasing subsequence which is not bounded above.

5. Let \((x_n)\) be a sequence such that the set \(S = \{x_n\} \) is finite. Show that there is a subsequence \((x_{n_i})\) such that for all \(i, x_{n_i} = x_{n_1}\).

**Theorem 7** Every bounded sequence contains a convergent subsequence.

**Proof:** Let \((x_n)_{n \geq 1}\) be a bounded sequence, and let \(S = \{x_n \mid n \geq 1\}\). Since the sequence is bounded there are numbers \(a_1 < b_1\) such that

\[
S \subset [a_1, b_1].
\]
We first construct by induction on \( n \) a decreasing sequence of intervals
\[
[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \cdots
\]
with the following two properties:

1. Each interval \([a_n, b_n]\) contains \(x_k\) for infinitely many \(k\), and

2. 
\[
b_n - a_n = \frac{b_1 - a_1}{2^{n-1}}. 
\]

By hypothesis, \(x_k \in [a_1, b_1]\) for all \(k\).

Now suppose by induction that the intervals \([a_i, b_i]\) are constructed for \(i \leq n\).
By construction \([a_n, b_n]\) contains \(x_k\) for infinitely many \(k\).
Then there are two mutually exclusive possibilities: either
\[
[a_n, \frac{a_n + b_n}{2}] \text{ contains } x_k \text{ for infinitely many } k,
\]
or
\[
[a_n, \frac{(a_n + b_n)}{2}] \text{ contains } x_k \text{ for only finitely many } k.
\]
In the second case, 
\[
\frac{a_n + b_n}{2}, b_n
\]
must contain \(x_k\) for infinitely many \(k\).

In the first case set
\[
a_{n+1} = a_n \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2}.
\]
In the second case set
\[
a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = b_n.
\]
By our induction hypothesis \(b_n - a_n = (b_1 - a_1)/2^{n-1}\), and therefore
\[
b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2} = \frac{b_1 - a_1}{2^n}.
\]
This completes the inductive construction.

Our second step is to construct by induction a subsequence \((x_{n_i})\) so that for all \(i\):
\[
x_{n_i} \in [a_i, b_i].
\]
First, set \(x_{n_1} = x_1\). Then
\[
x_{n_1} = x_1 \in S \subset [a_1, b_1]
\]
Then suppose by induction that the $x_{n_i}$ are constructed for $i \leq r$.
Since $[a_{r+1}, b_{r+1}]$ contains $x_k$ for infinitely many $k$ there is a least integer $p > n_r$
such that $x_p \in [a_{r+1}, b_{r+1}]$.
Set $x_{n_{r+1}} = x_p$.
This completes the inductive construction of the subsequence.

It remains to show that this subsequence is convergent.
Since
\[
 b_r - a_r = \frac{b_1 - a_1}{2^{r-1}}.
\]
it follows that the sequence $b_r - a_r$ converges to zero. Thus by the Corollary to
the Theorem, since $x_{n_r} \in [a_r, b_r]$, the sequence $(x_{n_r})$ converges. \textbf{q.e.d.}
Chapter 6

Continuous Functions of a Real Variable

6.1 Real-valued Functions of a Real Variable

Functions from the reals to the reals are a central tool in almost every discipline that uses mathematics. Among the many examples are:

- Position and speed of a particle as a function of time.
- Crop yield as a function of total precipitation.
- Grade on an exam as a function of time spent studying.
- Cost to the national health system as a function of the amount of pollution in the air.
- Fraction of the population that is illiterate as a function of the average time per pupil spent in class on reading and writing.
- Wave length of light reaching us from a star as a function of its distance away.

As you may easily imagine, functions of a real variable are also a core part of mathematics itself. In fact this field is a prime example of how mathematics interacts with other disciplines: many of the problems and theorems in mathematics in this area are inspired by questions and phenomena from outside, while the results and techniques that mathematicians discover frequently get applied elsewhere.
This chapter focuses on two key concepts in the analysis of functions: limits and continuity. But first we establish some basic definitions.

Definition 32 Intervals

1. An interval is a subset of \( \mathbb{R} \) of one of the following forms (where \( a \) and \( b \) are any real numbers):
   
   \( \begin{align*}
   & (a) \quad \mathbb{R} \\
   & (b) \quad [a, b], \ (a, b], \ [a, b) \ or \ (a, b) \\
   & (c) \quad [a, \infty), \ (a, \infty), \ (\infty, b] \ or \ (\infty, b) 
   \end{align*} \)

2. The intervals in (b) are called finite; the first is closed, the next two are half closed, and the fourth is open.

3. The numbers \( a \) and \( b \) are called the end points of their intervals.

4. The closure of an interval \( D \) is the union of \( D \) together with any end points. It is denoted by \( \overline{D} \).

Definition 33 A real-valued function is a set map

\[
f : D \to S
\]

from an interval \( D \) to a subset \( S \subseteq \mathbb{R} \). \( D \) is called the domain of \( f \) and \( S \) is its target.

Definition 34 If \( f : D \to S \) is a real-valued function and if \( E \) is a second interval contained in \( D \) then the restriction of \( f \) to \( E \) is the real function \( g : E \to S \) defined by \( g(x) = f(x), \ x \in E \).

Example 26 Real-valued functions

1. If \( f : D \to S \) and \( g : D \to S \) are real-valued functions then \( f + g : D \to \mathbb{R} \) and \( fg : D \to \mathbb{R} \) are the functions defined by

   \( (f + g)(x) = f(x) + g(x) \) and \( (fg)(x) = f(x)g(x) \).

2. If \( f : D \to S \) is a real-valued function and if \( f(x) \neq 0 \) for all \( x \in D \) then \( 1/f : D \to \mathbb{R} \) is the function defined by

   \( (1/f)(x) = 1/f(x) \).

3. \( f : \mathbb{R} \to \mathbb{R} \) defined by

   \[
f(x) = \begin{cases} 
   0, & x \in \mathbb{Q}, \\
   1, & x \notin \mathbb{Q}.
   \end{cases}
\]
4. \( f : (0, \infty) \rightarrow \mathbb{R} \) defined by: \( f(x) = \frac{1}{x} \).

5. \( f : (0, 1) \rightarrow \mathbb{R} \) defined by: \( f(x) \) is the number in the 25th decimal place of \( x \).

6. \( f(x) = \begin{cases} 
0, & x < 0, \\
1, & x \geq 0.
\end{cases} \)

7. \( f(x) = \sum_{i=0}^{n} \lambda_k x^k \), where the \( \lambda_k \) are real numbers and \( \lambda_n \neq 0 \). Such a function is called a polynomial of degree \( n \).

### 6.2 Limits

Suppose \( f : D \rightarrow S \) is a real-valued function. An important question which arises is:

**Can we use the values of \( f(x) \) near a point \( c \in \overline{D} \) to determine a ”limiting value”?**

There is a good practical reason for this question. In practice one can never make an exact measurement at a specific point, \( c \). The best one can do is make measurements nearby and hope they give a good approximation to a measurement at \( c \) itself.

This concept has **two fundamental ingredients**: 

1. We consider only values of the function near \( c \) and **not the value at** \( c \).

2. The values of the function at points near \( c \) must **bunch ever more closely together** as the points get closer to \( c \).

This idea is formalized in the following way:

**Definition 35** Suppose \( f : D \rightarrow S \) is a real-valued function defined in an interval \( D \), and that \( c \in \overline{D} \). Then

\[ f(x) \rightarrow u \text{ as } x \rightarrow c \]

if for any \( \varepsilon > 0 \) there is some \( \delta > 0 \) such that

\[ x \in D \text{ and } 0 < |x - c| < \delta \Rightarrow |f(x) - u| < \varepsilon. \]

**Note:** If you are asked to prove that \( f(x) \rightarrow u \) as \( x \rightarrow c \) then you are **given** some arbitrary positive \( \varepsilon \) about which **all you know** is that it is positive and then you have to **show** there is some positive \( \delta \) such that the condition holds. Usually \( \delta \) will **depend** on \( \varepsilon \).
Lemma 13 Suppose $f : D \to S$ is a real-valued function defined in an interval $D$. If for some $c \in D$, as $x \to c$

$$f(x) \to u_1 \quad \text{and} \quad f(x) \to u_2,$$

then $u_1 = u_2$.

Proof: Fix $\varepsilon > 0$. Choose $\delta_1 > 0$ and $\delta_2 > 0$ so that for $x \in D$,

$$|f(x) - u_1| < \varepsilon/2 \quad \text{if} \quad 0 < |x - c| < \delta_1,$$

and

$$|f(x) - u_2| < \varepsilon/2 \quad \text{if} \quad 0 < |x - c| < \delta_2.$$

Set $\delta$ to be the lesser of $\delta_1$ and $\delta_2$. Then for $x \in D$ and $0 < |x - c| < \delta$ we have

$$|u_2 - u_1| = |u_2 - f(x) + f(x) - u_1| \leq |u_2 - f(x)| + |f(x) - u_1| < \varepsilon.$$

Thus $|u_2 - u_1| < \varepsilon$ for all $\varepsilon > 0$ and so $u_2 = u_1$. q.e.d.

Definition 36 If $f(x) \to u$ as $x \to c$, then $u$ is the limit of $f(x)$ as $x$ approaches $c$ and we write

$$\lim_{x \to c} f(x) = u.$$

Important Remarks:

1. The definition of limit specifies the condition

$$0 < |x - c| < \delta.$$ 

Thus the limit depends on the values of $f(x)$ when $x$ is close to but different from $c$. Even when $c$ is in the interval $D$ the value of $f(c)$ is irrelevant to the definition of $\lim_{x \to c} f(x)$.

2. If $f : D \to S$ is a real-valued function and $c \in \overline{D}$, then there are two possibilities, and either could be correct:

(a) The limit of $f(x)$ as $x \to c$ exists, or

(b) This limit does not exist.

3. If $f : D \to S$ is a real-valued function and $c \in D$ then there are three possibilities:

(a) The limit of $f(x)$ as $x \to c$ exists and $\lim_{x \to c} f(x) = f(c)$, or

(b) The limit of $f(x)$ as $x \to c$ exists and $\lim_{x \to c} f(x) \neq f(c)$, or

(c) The limit of $f(x)$ as $x \to c$ does not exist.

Example 27 Limits

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1. \( D = \mathbb{R}, S = \mathbb{R} \) and \( f(x) = 5x + 1 \). Then

\[
\lim_{x \to 1} f(x) = 6.
\]

In fact given \( \varepsilon > 0 \) set \( \delta = \varepsilon/5 \). If \( 0 < |x - 1| < \delta \), then

\[
|f(x) - 6| = 5|x - 1| < 5(\varepsilon/5) = \varepsilon.
\]

2. \( D = S = \mathbb{R} \) and

\[
f(x) = \begin{cases} 
5x + 1, & x \neq 1, \\
17, & x = 1.
\end{cases}
\]

Then, as above, \( \lim_{x \to 1} f(x) = 6 \), but \( f(1) = 17 \).

3. \( D = (-\infty, 1), S = \mathbb{R} \) and \( f(x) = 5x + 1 \). Then

\[
\lim_{x \to 1} f(x) = 6.
\]

(Same proof as above.)

**Review of Negations:**

The **negation** of the statement "A is true" is the statement "A is false".

In analysis we frequently prove a statement is true by showing that the negation of the statement leads to a contradiction, and for this it is often helpful to restate the negation in a more useful form. Here is a typical example:

**Statement A:** \( \lim_{x \to c} f(x) = u \).

So what does it mean for Statement A to be false?

If Statement A is true then for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) satisfying the requirements of the definition above.

Therefore to say Statement A is false is precisely to say there must be some \( \varepsilon > 0 \) so that no \( \delta > 0 \) will work for that particular \( \varepsilon \).

Now what does it mean for \( \delta \) to work for that \( \varepsilon \)?

It means precisely that for every \( x \in D \) such that \( 0 < |x - c| < \delta \) it is true that \( |f(x) - u| < \varepsilon \).

Thus to say that no \( \delta > 0 \) will work for that \( \varepsilon \) means that for each \( \delta > 0 \) there is some \( x \in D \) such that

\[
0 < |x - c| < \delta \quad \text{and} \quad |f(x) - u| \geq \varepsilon.
\]
 Altogether then we have:

**Statement A** is false if and only if Statement B below is true:

**Statement B:** for some \( \varepsilon > 0 \) and every \( \delta > 0 \) there is some \( x \in D \) such that \( 0 < |x - c| < \delta \) and \( |f(x) - u| \geq \varepsilon \).

Statement B is the negation of Statement A.

**Example 28** \( D = S = R \) and

\[
f(x) = \begin{cases} 
1/|x|, & x \neq 0, \\
0, & x = 0.
\end{cases}
\]

In this example \( \lim_{x \to 0} f(x) \) does not exist.

We have to prove that no \( u \in R \) is the limit of \( 1/|x| \) as \( x \to 0 \). Fix any \( u \in R \).

We have to show that there is some \( \varepsilon > 0 \) with the following property:

For every \( \delta > 0 \) there is some \( x \in R \) such that

\[
0 < |x| < \delta \quad \text{and} \quad |1/|x| - u| > \varepsilon.
\]

In this example we choose \( \varepsilon = 1 \). Then, for any \( \delta > 0 \) choose \( x \) so that

\[
|x| < \delta \quad \text{and} \quad 0 < |x| < \frac{1}{|u| + 1}.
\]

Then \( 1/|x| > |u| + 1 \). Thus

\[
|1/|x| - u| \geq 1/|x| - |u| > 1 + |u| - |u| \geq 1 = \varepsilon.
\]

Therefore \( f(x) \) does not converge to \( u \) as \( x \to 0 \). Since \( u \) was any real number, the limit does not exist.

In the previous chapter we introduced the limit of an infinite sequence, and now we have another kind of limit. It is natural to wonder if these two kinds of limit are connected, and indeed they are. The next Proposition shows how.

**Proposition 12** Let \( f : D \to S \) be a real-valued function defined in an interval \( D \), and suppose \( c \in \overline{D} \). Then

\[
\lim_{x \to c} f(x) = u
\]

if and only if

\[
\lim_{n} f(x_n) = u
\]

whenever \( (x_n) \) is a sequence of points in \( D \) different from \( c \), but convergent to \( c \).
Proof: We have two prove two things:

1. If \( \lim_{x \to c} f(x) = u \) then \( \lim_n f(x_n) = u \) whenever \( (x_n) \) is a sequence of points in \( D \) different from \( c \) but convergent to \( c \).

2. If \( \lim_n f(x_n) = u \) whenever \( (x_n) \) is a sequence of points in \( D \) different from \( c \) but convergent to \( c \), then \( \lim_{x \to c} f(x) = u \).

For the first statement our hypothesis is \( \lim_{x \to c} f(x) = u \).

Then if \( (x_n) \) is any sequence of points in \( D \) different from but converging to \( c \), we need to prove that

\[
\lim_n f(x_n) = u.
\]

In other words, given \( \varepsilon > 0 \) we need to show that for some \( N \),

\[
|f(x_n) - u| < \varepsilon \quad \text{if} \quad n \geq N.
\]

Since by hypothesis \( \lim_{x \to c} f(x) = u \), it follows that for some \( \delta > 0 \),

\[
|f(x) - u| < \varepsilon \quad \text{if} \quad 0 < |x - c| < \delta.
\]

Since the sequence \( (x_n) \) converges to \( c \) and no \( x_n = c \) there is some \( N \in \mathbb{N} \) for which

\[
0 < |x_n - c| < \delta \quad \text{if} \quad n \geq N.
\]

Therefore \( |f(x_n) - u| < \varepsilon \) if \( n \geq N \). This proves that the sequence \( f(x_n) \) converges to \( u \).

For the second statement our hypothesis is that whenever \( (x_n) \) is a sequence of points in \( D \) different from but converging to \( c \) then

\[
\lim_n f(x_n) = u,
\]

and we need to prove that

\[
\lim_{x \to c} f(x) = u.
\]

We prove this by showing that the negation of this statement is false, and so the statement must be true.

As described above the negation is Statement B:

For some \( \varepsilon > 0 \) and every \( \delta > 0 \) there is some \( x \in D \) such that:

\[
0 < |x - c| < \delta \quad \text{and} \quad |f(x) - u| \geq \varepsilon.
\]

We prove this is false by contradiction: we assume Statement B is true and deduce a contradiction.
Now if Statement B is true, then for each \( n \in \mathbb{N} \) there is some point \( x_n \in D \) such that
\[
0 < |x_n - c| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - u| \geq \varepsilon.
\]
This defines a sequence \((x_n)\) of points in \( D \) different from \( c \). Moreover, for any real number \( \sigma > 0 \) there is some \( N \in \mathbb{N} \) such that \( 1/N < \sigma \) (Property Three for the real numbers).

Therefore, for \( n \geq N \)
\[
|x_n - c| < 1/n \leq 1/N < \sigma.
\]
Thus the sequence \((x_n)\) converges to \( c \).

Now our hypothesis states that since \((x_n)\) converges to \( c \), it follows that \( f(x_n) \) converges to \( u \). But we constructed the sequence so that
\[
|f(x_n) - u| \geq \varepsilon
\]
for all \( n \), which contradicts this hypothesis.

It follows that Statement B cannot be correct and therefore that \( \lim_{x \to c} f(x) = u \).

q.e.d.

**Exercise 19 Limits**

1. If \( c < d \) are points in an interval \( D \), show that \([c, d] \subseteq D\).

2. Suppose \( f : (a, b) \to \mathbb{R} \) is a real-valued function and that for some \( u \in \mathbb{R} \), \( f(x) < u \) for all \( x \in (a, b) \). If \( \lim_{x \to a} f(x) \) exists, show that
\[
\lim_{x \to a} f(x) \leq u.
\]

3. For each of the following choices of \( D \), \( f \), and \( c \), and with \( S = \mathbb{R} \) decide if \( \lim_{x \to c} f(x) \) exists and when it does find the limit.
   
   (a) \( D = (-1, 0) \), \( c = 0 \), \( f(x) = 1 \), \( x \in D \).

   (b) \( D = [0, 1) \), \( c = 0 \), and \( f(x) = 0 \), \( x \in D \).

   (c) \( D = (-1, 1) \), \( c = 0 \), and
   \[
   f(x) = \begin{cases} 
   1, & x \in (-1, 0), \\
   0, & x \in (0, 1).
   \end{cases}
   \]

4. Suppose \( f : D \to \mathbb{R} \) and \( g : D \to \mathbb{R} \) are real-valued functions in an interval \( D \). If \( c \in D \) and if \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) exist, show that
\[
\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x), \quad \text{and}
\]
\[
\lim_{x \to c} f(x) g(x) = (\lim_{x \to c} f(x)) (\lim_{x \to c} g(x)).
\]

In particular conclude that these limits exist.
5. Suppose $f : D \to S$ is a real-valued function in an interval $D$. Suppose further that for some $c \in D$ and some $\alpha > 0$,

$$f(x) \neq 0 \text{ if } 0 \leq |x - c| < \alpha.$$ 

Show that

$$\lim_{x \to c} \frac{1}{f(x)}$$

exists if $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) \neq 0$.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a polynomial. Show that for all $y \in \mathbb{R}$,

$$\lim_{x \to y} f(x) = f(y).$$

7. Suppose $z \in \mathbb{R}$, $f : D \to \mathbb{R}$ is a real-valued function in an interval, and $c \in D$. Assume that for each $0 < \varepsilon < 1$ there is a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - z| < \varepsilon$. Show that $\lim_{x \to c} f(x) = z$.

8. Suppose $x, y$ are positive real numbers and $n \in \mathbb{N}$.

(a) If $n \geq 2$ use the Difference theorem to show that $|x^{1/n} - y^{1/n}| < \frac{|x - y|}{y^{(n-1)/n}}$.

(b) Show that $\lim_{x \to y} x^{1/n} = y^{1/n}$.

9. Do the following limits exist?

$$\lim_{x \to 0} \sin \left( \frac{1}{x} \right) \text{ and } \lim_{x \to 0} x \sin \left( \frac{1}{x} \right).$$

You may use the following facts:

(a) $-1 \leq \sin(y) \leq 1$ for all $y \in \mathbb{R}$.

(b) 

$$\sin \left( \frac{\pi k}{2} \right) = \begin{cases} 
0, & k = 2n, \\
1, & k = 4n + 1, \\
-1, & k = 4n - 1.
\end{cases}$$

10. Define a real-valued function $f : \mathbb{R} \to \mathbb{R}$ in the interval by

$$f(x) = \begin{cases} 
0, & x < 0, \\
1, & x \geq 0.
\end{cases}$$

For which $c \in \mathbb{R}$ does $\lim_{x \to c} f(x)$ exist?

11. Let $n \in \mathbb{N}$ and define a function $f : (-\infty, a) \to \mathbb{R}$ by

$$f(x) = \frac{x^n - a^n}{x - a}.$$ 

Show that $\lim_{x \to a} f(x)$ exists, find the limit and prove your answer is correct.
6.3 Continuous Functions

Let \( f : D \to S \) be a real-valued function defined in an interval, \( D \). If \( c \in D \) and if \( \lim_{x \to c} f(x) \) exists then we have two numbers:

\[
\lim_{x \to c} f(x) \quad \text{and} \quad f(c).
\]

As we saw in the previous exercises, these two numbers may be different!

However, if these numbers are the same then we can approximate the value at \( c \) of \( f \) by the values nearby \( c \). Functions for which these two numbers are the same have many good properties, and play a crucial role in mathematics. In fact they have a name:

**Definition 37** A real-valued function \( f : D \to S \) in an interval \( D \) is continuous if for each \( c \in D \) the limit \( \lim_{x \to c} f(x) \) exists and

\[
\lim_{x \to c} f(x) = f(c).
\]

**Lemma 14** Suppose \( f : D \to S \) is a continuous function in an interval \( D \). Then for each \( \varepsilon > 0 \) and \( c \in D \) there is a \( \delta > 0 \) such that

\[
x \in D \quad \text{and} \quad |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon.
\]

**Remark:** The \( \delta \) in the Lemma will usually depend on both \( \varepsilon \) and \( x \).

**Proof:** Since \( f \) is continuous, \( \lim_{x \to c} f(x) = f(c) \). In particular, the limit exists.

Fix any \( \varepsilon > 0 \).

By the definition of limit, there is some \( \delta > 0 \) such that if \( x \in D \) and \( 0 < |x - c| < \delta \), then

\[
|f(x) - f(c)| < \varepsilon.
\]

But this inequality is trivially true if \( x = c \).

q.e.d.

**Lemma 15** A function \( f : D \to S \) in an interval \( D \) is continuous if and only if whenever a sequence \( (x_n) \) of points in \( D \) converges to \( c \in D \) then the sequence \( (f(x_n)) \) converges to \( f(c) \).

**Remark:** This Lemma states that a function is continuous if and only if it preserves convergent sequences.

**Proof:** First, suppose \( f \) is continuous. Choose \( \delta > 0 \) so that

\[
|f(x) - f(c)| < \varepsilon \quad \text{if} \quad x \in D \quad \text{and} \quad |x - c| < \delta.
\]

Since the sequence \( (x_n) \) in our hypothesis converges to \( c \in D \), it follows that for some \( N \),

\[
|x_n - c| < \delta \quad \text{if} \quad n \geq N.
\]
Thus for \( n \geq N \),

\[ |f(x_n) - f(c)| < \varepsilon. \]

Therefore \((f(x_n)) \) converges to \( f(x) \).

On the other hand suppose \( f \) preserves convergent sequences. If \( c \in D \) then for any sequence \((x_n)\) of points in \( D \) converging to \( c \), by hypothesis

\[ \lim_{n \to \infty} f(x_n) = f(c). \]

It follows from the Proposition in the previous section that \( \lim_{x \to c} f(x) = f(c) \).

q.e.d.

**Exercise 20  Continuous functions**

1. Show that if \( \lambda \in \mathbb{R} \) the constant function \( \mathbb{R} \to \lambda \) is continuous.

2. If \( f : D \to S \) and \( g : D \to S \) are continuous functions in an interval \( D \), show that the functions \( f + g \) and \( fg \) are continuous.

3. If \( f : D \to S \) is a continuous function in an interval \( D \), and if \( f(x) \neq 0 \) for all \( x \in D \), show that \( 1/f \) is continuous.

4. Show that polynomials are continuous functions in \( \mathbb{R} \).

5. If \( n \in \mathbb{N} \) show that \( x^{1/n} \) is a continuous function in \((0, \infty)\).

6. Show that \( \sin x \) and \( \cos x \) are continuous functions. (You may use the trigonometric formulae for \( \sin(x + y) \) and \( \cos(x + y) \).)

7. If \( f : D \to \mathbb{R} \) is a continuous function in an interval \( D \) and if \( g : \mathbb{R} \to T \) is a continuous function in \( \mathbb{R} \), show that the composite \( g \circ f \) is continuous.

8. If \( f : D \to S \) is a continuous function defined in an interval \( D \), show that the restriction of \( f \) to a second interval contained in \( D \) is continuous.

9. If \( f : D \to S \) is a continuous function defined in \( D = [a, b) \) and if \( \lim_{x \to b^-} f(x) = u \), show that a continuous function \( g : [a, b] \to \mathbb{R} \) is defined by

\[
g(x) = \begin{cases} \quad f(x), & x \in [a, b), \\ u, & x = b. \end{cases}
\]

10. Suppose \( f : D \to S \) is a continuous function in an interval \( D \).

(a) Show that the function \( |f| : D \to \mathbb{R} \) defined by \( |f|(x) = |f(x)| \) is continuous.
(b) Show that the functions \( f_+ : D \to \mathbb{R} \) and \( f_- : D \to \mathbb{R} \) defined by

\[
\begin{align*}
  f_+(x) &= \begin{cases} 
    f(x), & f(x) \geq 0, \\
    0, & f(x) < 0.
  \end{cases} \\
  f_-(x) &= \begin{cases} 
    -f(x), & f(x) \leq 0, \\
    0, & f(x) > 0.
  \end{cases}
\end{align*}
\]

are continuous. (Hint: Consider \( 1/2(f + |f|) \).)

(c) Show that \( f = f_+ - f_- \).

Recall that a continuous function is a real-valued function whose value at a given point is the limit of the values at points approaching the given point. Continuous functions in a closed interval automatically satisfy a stronger condition:

**Proposition 13** Suppose \( f : D \to S \) is a continuous function defined in a closed interval \( D = [a, b] \). Then for all \( \varepsilon > 0 \) there is some \( \delta > 0 \) such that

\[ |f(y) - f(x)| < \varepsilon \quad \text{if} \quad x, y \in [a, b] \quad \text{and} \quad |x - y| < \delta. \]

**Remark:** Because \([a, b]\) is closed the \( \delta \) does not depend on \( x \).

**Proof of Proposition 13:** We assume the Proposition is false and derive a contradiction.

If the Proposition is false there is some \( \varepsilon > 0 \) such that for each \( \delta > 0 \) there is a corresponding pair of points \( x, y \in [a, b] \) satisfying

\[ |f(x) - f(y)| \geq \varepsilon \quad \text{and} \quad |x - y| < \delta. \]

In particular, we may find sequences \((x_n)\) and \((y_n)\) of points in \([a, b]\) such that

\[ |f(x_n) - f(y_n)| \geq \varepsilon \quad \text{and} \quad |x_n - y_n| < \frac{1}{n}. \]

Since the sequences \((x_n)\) and \((y_n)\) are bounded the sequence \((x_n)\) contains a convergent subsequence \((u_k) = (x_{n_k})\). In the same way the sequence \((v_k) = (y_{n_k})\) contains a convergent subsequence \((v_k)\). As a subsequence of a convergent sequence, the subsequence \((u_k)\) is also convergent.

Set

\[ \lim_{k} (u_k) = x \quad \text{and} \quad \lim_{k} (v_k) = y. \]

Since the \( x_n \) and the \( y_n \) are in \([a, b]\), it follows that \( x \in [a, b] \) and \( y \in [a, b] \).
Since \(|x_n - y_n| < 1/n\) it follows that
\[
\lim_{k} |u_{ik} - v_{ik}| = 0,
\]
and so \(x = y\).

Since \(f\) is continuous it preserves convergent sequences and so
\[
f(x) = \lim_{k} f(u_{ik}) = \lim_{k} f(v_{ik}) = f(y).
\]
But this is impossible because
\[
|f(y_n) - f(x_n)| \geq \varepsilon
\]
for all \(n\).

This is the desired contradiction.
q.e.d.

6.4 Continuous Functions Preserve Intervals

In this section we establish an important and fundamental property of continuous functions.

**Theorem 8** Suppose \(f : D \to S\) is a continuous function defined in a closed interval \(D = [a, b]\). Then the image of \(f\) is a closed interval:
\[
f([a, b]) = [c, d]
\]
for some real numbers \(c \leq d\).

Consequences of Theorem 8.

1. **Maximum and minimum values.** Since \([c, d]\) is the image of \(f\) there must be points \(u, v \in [a, b]\) such that
\[
f(u) = c \quad \text{and} \quad f(v) = d.
\]
Thus
\[
f(u) \leq f(x) \leq f(v)
\]
for all \(x \in [a, b]\).

2. **Intermediate values.** Since \([c, d]\) is the image of \(f\) it follows that for every \(y \in [c, d]\) there is some \(x \in [a, b]\) such that
\[
f(x) = y.
\]
**Important Remarks:** Let $f : D \to S$ be a continuous function defined in the interval $D = [a, b]$.

1. Then the Theorem states that
   
   (a) The set of values of $f$ is bounded.
   
   (b) $f$ has a minimum and maximum value.
   
   (c) Every number between the minimum and maximum value is a value of $f$.

2. However, the points $u, v$ where $f$ attains its minimum and maximum values may be anywhere inside the interval $[a, b]$, and it may well happen that $v < u$.

**Example 29**

1. $[a, b] = [-1, 1]$ and
   
   \[
   f(x) = \begin{cases} 
   x, & x \leq 0, \\
   -2x, & x \geq 0. 
   \end{cases}
   \]
   
   Here the image of $f$ is the interval $[-2, 0]$ and $f$ attains its minimum at 1 and its maximum at 0.

2. $f$ is defined in $(0, 1)$ by $f(x) = 1/x$. Here the values of $f$ are not bounded above but are bounded below by 1; however, $f$ does not attain a minimum value since $\text{glb}\{f(x)\} = 1$ and 1 is not a value of $f$.

While the properties above are consequences of the Theorem we actually have to prove them first and then use them to prove the Theorem! We do this now.

**Proposition 14** Suppose $f : D \to S$ is a continuous function defined in a closed interval $D = [a, b]$. Then for some $u, v \in [a, b]$,

\[
f(u) \leq f(x) \leq f(v)
\]

for all $x \in [a, b]$.

**Proof:** Set $S = \{f(x) \mid x \in [a, b]\}$. We show first by contradiction that $S$ is bounded above.

In fact, if $S$ is not bounded above then for each $n \in \mathbb{N}$ there is an $x_n \in [a, b]$ such that $f(x_n) > n$. Since $x_n$ is a bounded sequence it contains a convergent subsequence $(x_{n_i})$, and the limit $z$ of this subsequence is in $[a, b]$. Since continuous functions preserve convergent sequences the sequence $(f(x_{n_i}))$ is convergent. But

\[
f(x_{n_i}) > n_i
\]

and so this sequence is not bounded above. Therefore it cannot be convergent. This is a contradiction, and so $S$ must be bounded above.
Since $S$ is bounded above, for each $n$ there is some $y_n \in S$ such that

$$0 \leq \text{lub}(S) - y_n < \frac{1}{n}.$$  

In particular, $\lim_{n}(y_{n}) = \text{lub}(S)$.

On the other hand, because $y_n \in S$ we may write $y_n = f(x_n)$ for some $x_n \in [a, b]$. Then $(x_n)$ is a bounded sequence. Therefore there is a subsequence $x_{n_k}$ converging to some point $v \in [a, b]$. Because continuous functions preserve convergent sequences the sequence $f(x_{n_k})$ converges to $f(v)$.

But $f(x_{n_k})$ is a subsequence of the convergent sequence $f(x_n)$. Therefore this subsequence has the same limit. Hence

$$f(v) = \lim_{k} f(x_{n_k}) = \lim_{n} f(x_n) = \lim_{n} y_{n} = \text{lub}(S).$$

In other words, for all $x \in [a, b]$,

$$f(x) \leq f(v).$$

The proof of the existence of $u$ is identical.

q.e.d.

Definition 38 The points $u, v$ in Proposition 11 are called respectively an absolute minimum point and an absolute maximum point for $f$.

Proposition 15 If $f : D \to S$ is a continuous function defined in an interval $D$ and if $f(x) < f(y)$ for some $x, y \in D$, then for any $w$ with $f(x) < w < f(y)$ there is some $z \in D$ with $f(z) = w$.

Proof: We argue by contradiction, and so suppose that there is no $z \in D$ with $f(z) = w$.

Let $d = |y - x|$. We construct by induction a sequence of intervals $I_n \subset D$, $n \geq 0$ with endpoints $x_n$ and $y_n$ such that

$$|y_n - x_n| = \frac{d}{2^n}, \text{ and}$$

$$f(x_n) < w < f(y_n).$$

(Note: it may happen that $x_n < y_n$ or that $y_n < x_n$!!)

First set $x_0 = x$ and $y_0 = y$.

Now suppose $x_n$ and $y_n$ are constructed. Since $w$ is not a value, either

$$f\left(\frac{x_n + y_n}{2}\right) < w \text{ or } f\left(\frac{x_n + y_n}{2}\right) > w.$$
In the first case, set

\[ x_{n+1} = \frac{x_n + y_n}{2} \quad \text{and} \quad y_{n+1} = y_n. \]

In the second case, set

\[ x_{n+1} = x_n \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}. \]

In either case

\[ |y_{n+1} - x_{n+1}| = \frac{d}{2^{n+1}}, \quad \text{and} \quad f(x_{n+1}) < w < f(y_{n+1}). \]

Now by the Intersection Theorem, \( \bigcap I_n \) is a single point \( c \). Moreover, the Corollary to that Theorem states that

\[ \lim_{n \to \infty} x_n = c = \lim_{n \to \infty} y_n. \]

Since \( f \) is continuous, it follows that

\[ \lim_{n \to \infty} f(x_n) = f(c) = \lim_{n \to \infty} f(y_n). \]

But \( f(x_n) < w \) for all \( n \), and thus \( f(c) \leq w \). Also \( f(y_n) > w \) for all \( n \), and thus \( f(c) \geq w \).

It follows that \( f(c) = w \), contradicting our hypothesis that \( w \) was not a value of \( f \). \( \text{q.e.d.} \)

**Proof of Theorem 8:** By Proposition 14, \( f \) has an absolute minimum at some \( u \in [a, b] \) and an absolute maximum at some \( v \in [a, b] \):

\[ f(u) \leq f(x) \leq f(v) \]

for all \( x \in [a, b] \). If \( w \in [f(u), f(v)] \), then either \( w = f(u) \) or \( w = f(v) \) or \( f(u) < w < f(v) \). But in this third case it follows from Proposition 15 that \( w = f(x) \) for some \( x \in [a, b] \).

**Exercise 21**

In these exercises you may assume the standard properties of \( \sin x \) and \( \cos x \).

1. Let \( f : D \to S \) be any real-valued function defined in an interval \( D \).

   (a) Show that if \( f \) attains a maximum then that maximum is \( \text{lub}(S) \) where
   \[ S = \{ f(x) \mid x \in D \}. \]

   (b) If \( f \) attains a minimum value, what is it?