Please sign the honor pledge on the front of the booklet. You must carefully justify all of your statements. You may use anything from class or the book without proof.

1. (a) (5 pts.) State the Cauchy-Schwarz inequality.
(b) (15 pts.) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be continuously differentiable function, \( f(0) = 0 \). Suppose \( |f(u)| = 1 \) for some \( \|u\| = 1 \). Prove that there exists \( v \) such that \( \|\nabla f(v)\| \geq 1 \).

**Answer:**

(a) For all vectors \( x, y \in \mathbb{R}^n \),
\[
|\langle x, y \rangle| \leq \|x\| \cdot \|y\|.
\]

It is not correct if you just write the inequality without explaining what \( x \) and \( y \) are!!

(b) Since \( f \) is \( C^1 \) on \( \mathbb{R}^n \), by the Mean Value Theorem (Thm 13.17) there exists \( 0 < \theta < 1 \) such that
\[
f(0 + u) - f(0) = \langle \nabla f(0 + \theta u), u \rangle.
\]

Since \( f(0) = 0 \) and \( |f(u)| = 1 \), by Cauchy-Schwarz,
\[
1 = |f(u)| = |f(0 + u) - f(0)| = |\langle \nabla f(0 + \theta u), u \rangle| \leq \|\nabla f(0 + \theta u)\| \cdot \|u\| = \|\nabla f(\theta u)\|
\]

Let \( v = \theta u \).

2. (20 pts.) Prove that
\[
\sin xy + e^{xz} = 1 \\
5y^2 - 3z^2 + x^2 = 2
\]
implicitly defines \( x \) and \( y \) as a \( C^1 \) function \( \psi(z) \) near the solution \((0, 1, 1)\), and find the best affine approximation of \( \psi \) at \( z = 1 \).

**Answer:** Let \( \mathbf{F}(x, y, z) = (\sin xy + e^{xz}, 5y^2 - 3z^2 + x^2) \). \( \mathbf{F} \) is easily seen to be continuously differentiable. Further
\[
\mathbf{D}\mathbf{F}(x, y, z) = \begin{bmatrix}
y \cos xy + ze^{xz} \\
2x \\
2x \cos xy \\
10y \\
x e^{xz} \\
-6z
\end{bmatrix}
\]

so
\[
\mathbf{D}\mathbf{F}(0, 1, 1) = \begin{bmatrix}
2 & 0 & 0 \\
0 & 10 & -6
\end{bmatrix}
\]

The left \( 2 \times 2 \) minor is nonsingular, therefore by the Implicit Function Theorem there exists \( \delta > 0 \) and a continuously differentiable function \( \psi : B_\delta(1) \to \mathbb{R}^2 \) such that the solutions to the system \( \mathbf{F}(x, y, z) = (1, 2) \) are implicitly defined by \( (x, y) = \psi(z) \), for \((x, y) \in B_\delta(0, 1)\). Further
\[
D\psi|_{z=1} = -\begin{bmatrix}
2 & 0 \\
0 & 10
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
-6
\end{bmatrix} = -\begin{bmatrix}
1/2 & 0 \\
0 & 1/10
\end{bmatrix} \begin{bmatrix}
0 \\
-6
\end{bmatrix} = \begin{bmatrix}
0 \\
3/5
\end{bmatrix}
\]
Thus the best affine approximation to $\psi$ at $z = 1$ is
$$\psi(1 + h) \approx \psi(1) + D\psi|_{z=1} h = (0, 1 + 3/5 \cdot h).$$

3. (20 pts.) Let $n > 0$ be even. Maximize the function $f(x, y, z) = \left(\frac{x+y+z}{3}\right)^n$ subject to the constraint $x^n + y^n + z^n = 3$. Conclude that for all real numbers $a, b, c$,
$$\frac{a + b + c}{3} \leq \left(\frac{a^n + b^n + c^n}{3}\right)^{1/n}.$$

**Answer:** Let $g(x, y, z) = x^n + y^n + z^n$ be the constraint function. Notice that the constraint set $g(x, y, z) = 3$ is compact: $g$ is continuous and level sets of continuous functions are closed, and the fact that $n$ is even implies the level set is bounded. Thus $f$ attains a maximum on the constraint set, say at $(x_0, y_0, z_0)$.

Now
$$\nabla f(x, y, z) = \left(\frac{n}{3} \left(\frac{x+y+z}{3}\right)^{n-1} \cdot \left(\frac{x+y+z}{3}\right)^{n-1} \cdot \left(\frac{x+y+z}{3}\right)^{n-1}\right)$$

$$\nabla g(x, y, z) = (nx^{n-1}, ny^{n-1}, nz^{n-1})$$

Both functions are polynomials, so $C^1$. The condition $\nabla g(x, y, z) = (0, 0, 0)$ implies $(x, y, z) = (0, 0, 0)$ which cannot happen given the constraint. Therefore we can apply the Lagrange Multiplier Theorem and say there exists $\lambda$ such that
$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0).$$

If $\lambda = 0$ then $(x + y + z)/3 = 0$ which gives the minimum value of $f(x, y, z) = 0$.
Otherwise, since $\nabla f(x_0, y_0, z_0) = * (1, 1, 1)$ for some $*$, this implies $nx_0^{n-1} = ny_0^{n-1} = nz_0^{n-1}$, therefore $x_0 = y_0 = z_0$ since $n - 1$ is odd. Plugging back into the constraint gives $(x_0, y_0, z_0) = \pm(1, 1, 1)$, which gives the maximum value, 1, of the function $f$. That is
$$f(x, y, z) \leq 1 \quad \text{for all } x^n + y^n + z^n = 3.$$

Now if $a, b, c$ are arbitrary real numbers set $S = ((a^n + b^n + c^n)/3)^{1/n}$, and $x = a/S$, $y = b/S$, $z = c/S$. Then $(x, y, z)$ satisfies the constraint; we conclude that $f(x, y, z) \leq 1$.

Writing this equation in terms of $a, b, c$ and multiplying through by $S^n$ and taking the $n$th root gives the result.

4. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a continuously differentiable function. Define the solution set
$$S = \{x \mid g(x) \leq c\}.$$

Suppose that $\nabla g(x) \neq 0$ for all $x \in S$.

(a) (5 pts.) Prove that $\partial S = \{x \mid g(x) = c\}$ (Hint: Prove $\subset$ then $\supset$).
5. (a) (5 pts.) Let $x$ be a generalized rectangle in $\mathbb{R}$. Assume that $f(x) \geq 0$ for all points $x$ in $I$. Prove that if $\int_I f = 0$, then $f(x) = 0$ for all points $x$ in $I$.

(b) (5 pts.) Prove that for every $x \in \partial S$, there exists an open set $U \ni x$ such that $\partial S \cap U$ has Jordan content 0. It follows that if $\partial S$ is bounded, then it has Jordan content 0 (don’t prove this).

(c) (5 pts.) Define what it means for a set to have volume. Explain why $B_1(0,0,0)$ has volume and set up (but do not evaluate) an iterated integral that computes it.

(d) (5 pts.) Find a bounded set $A \subset \mathbb{R}^3$, and a function $f : A \to \mathbb{R}$ such that $f$ is continuous on $A$, but $f$ is not integrable.

Answer:

(a) ($\subseteq$) Assume $x \in \partial S$. By definition, for every $k > 0$ there exists $u_k, v_k \in B_k(x)$ such that $u_k \in S, v_k \not\in S$. In other words, $g(u_k) \leq c$ and $g(v_k) > c$. $g$ is $C^1$, so continuous, so by comparison,

$$g(x) = \lim_{k \to \infty} g(u_k) \leq c, \quad \text{and,} \quad g(x) = \lim_{k \to \infty} g(v_k) \geq c.$$ 

We conclude that $g(x) = c$. 

($\supseteq$) Assume $g(x) = c$. The condition $\nabla g(x) \neq 0$ implies that $x$ is not a relative extremum. In other words, for every $\varepsilon > 0$ there exists $u, v \in B_\varepsilon(x)$ such that $g(u) \geq g(x) = c$ and $g(v) \leq g(x) = c$. Thus $u \in S$ and $v \not\in S$; since $\varepsilon$ was arbitrary we conclude that $x \in \partial S$.

(b) Let $x = (x, y, z)$. Since $\nabla g(x, y, z) \neq 0$ we may assume after relabeling that $D_3 g(x, y, z) \neq 0$. Thus we can apply the Implicit Function Theorem to say there is $\delta > 0$ and a $C^1$ function $\psi : B_\delta(x, y) \to \mathbb{R}$ such that for $z' \in B_\delta(z)$

$$g(x', y', z') = c \iff z' = \psi(x', y').$$

Let $U = B_\delta(x, y) \times B_\delta(z) \subset \mathbb{R}^3$. The previous statement exactly says that $\partial S \cap U$ is exactly the graph of the function $\psi$. $\psi$ is $C^1$, so continuous, and integrable on the Jordan domain $B_\delta(x, y)$. Thus $\partial S \cap U = \text{(graph of } \psi)$ has Jordan content 0 (Cor 18.26).

(c) A set $S$ is said to have volume if the function $f : S \to \mathbb{R}$ which is constantly 1 is integrable. By parts (a) and (b), $B_1(0,0,0)$ is a Jordan domain because it is of the form given with $g(x, y, z) = x^2 + y^2 + z^2$ and $c = 1$. Following Example 19.11 in the book, the volume of the ball is given by the integral $\int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \int_{r=0}^{1} r^2 \sin \theta \, dr \, d\varphi \, d\theta$.

(d) The exam should have said “bounded set”. Certain if $A$ is not bounded then no function on $A$ is integrable, just because we never defined what that meant. An example of a bounded set $A$ is the set of points in $I = [0,1] \times [0,1] \times [0,1]$ with a rational component, and $f$ is the function that is constantly 1 on $A$. If $\hat{f}$ is the zero extension of $f$ to $I$, then $M(\hat{f}, J) = 1$ and $m(\hat{f}, J) = 0$ for every subrectangle $J \subset I$, by the density of the rationals and irrationals in $\mathbb{R}$. It follows that $U(\hat{f}, P) = 1$ and $L(\hat{f}, P) = 0$ for every partition of $I$, so $f$ is not integrable because $\hat{f}$ is not.
(b) (10 pts.) Define \( f : [0, 1] \times [0, 1] \rightarrow \mathbb{R} \)

\[
f(x, y) = \begin{cases} 
1 & x = y \\
0 & \text{otherwise}
\end{cases}
\]

Prove that \( \int_{[0,1] \times [0,1]} f = 0 \) using the Archimedes-Riemann theorem.

**Answer:**

(a) Prove the contrapositive. Assume \( f(x_0) > 0 \) for some \( x_0 \in I \) (say \( f(x_0) = \varepsilon \)). To simplify things, by continuity we may assume that \( x_0 \) is in the interior of \( I \). By continuity there exists \( \delta \) such that \( B_\delta(x_0) \subset I \) and if \( x \in B_\delta(x_0) \) then \( |f(x) - f(x_0)| < \varepsilon/2 \) (i.e., \( f(x) > \varepsilon/2 \)). Define \( g : I \rightarrow \mathbb{R} \) by

\[
g(x) = \begin{cases} 
\varepsilon/2 & x \in B_\delta(x_0) \\
0 & \text{otherwise}
\end{cases}
\]

By construction, \( f(x) \geq g(x) \) for all \( x \in I \). We need to mention that \( g \) is integrable because it is the a multiple of the characteristic function of the Jordan domain \( B_\delta(x_0) \). By monotonicity, \( \int_I f \geq \int_I g = \varepsilon/2 \cdot \text{vol} B_\delta(x_0) > 0 \).

(b) Let \( P_k \) be the standard partition of \( I = [0, 1] \times [0, 1] \) into \( k^2 \) equally sized squares. These squares fall into two types: those \( J \) that intersect the diagonal and those \( J' \) that don’t. There are exactly \( k + 2(k - 1) = 3k - 1 \) squares that intersect the diagonal (I don’t mind if you say there are \( k \)). Since \( f \) is constantly 0 on the squares \( J' \),

\[
M(f, J) = 1 \\
m(f, J) = 0 \\
M(f, J') = m(f, J') = 0
\]

so

\[
U(f, P_k) = \sum_J 1 \cdot \text{vol} J + \sum_J 0 \cdot \text{vol} J = (3k - 1) \cdot \frac{1}{k^2} \xrightarrow{k \to \infty} 0
\]

\[
L(f, P_k) = \sum_J 0 \cdot \text{vol} J + \sum_J 0 \cdot \text{vol} J = 0 \xrightarrow{k \to \infty} 0
\]

Since these limits agree, \( P_k \) is an Archimedean sequence of partitions; by the Archimedes-Riemann theorem, \( f \) is integrable and \( \int_I f = \lim_{k \to \infty} U(f, P_k) = 0 \).