Math 411 — Homework 11— Due July 13, 2010

49. Exercises 2 & 11, Section 18.1

50. Exercise 9, Section 18.2

51. In both parts, \( I \) is a generalized rectangle in \( \mathbb{R}^n \) and \( f : I \rightarrow \mathbb{R} \) is a bounded function.

(a) Give an example of a function \( f \) such that \( |f| \) is integrable over \( I \) but \( f \) is not.\(^1\)

(b) Define new functions \( f^+, f^- : I \rightarrow \mathbb{R} \) by the rules

\[
  f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases} \\
  f^-(x) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ f(x) & \text{if } f(x) < 0 \end{cases}
\]

Prove that

\[
  M(f^+, I) - m(f^+, I) \leq M(f, I) - m(f, I), \quad \text{and}, \\
  M(f^-, I) - m(f^-, I) \leq M(f, I) - m(f, I)
\]

(c) Prove that if \( f \) is integrable over \( I \), then \( |f| \) is integrable over \( I \) (Hint: find an equation relating the functions \( f, |f|, f^+, \) and \( f^- \))

52. Recall that in the definition of Jordan content 0, we required a set to be covered by finitely many generalized rectangles. Suppose we change the definition to allow any infinite sequence of generalized rectangles, and call this \( \sigma \)-Jordan content 0.

(a) Prove that if a set is countable, then it has \( \sigma \)-Jordan content 0 (Hint: find a series such that \( \sum_{n=1}^{\infty} a_n = 1 \)).

(b) Prove that the set of rational numbers has \( \sigma \)-Jordan content 0.

\(^1\)|\( f | \) is the function \( I \rightarrow \mathbb{R} \) defined by the rule \( |f(x)| = |f(x)| \)