40. Exercise 14, Section 17.1  

**Answer:** (a) By the Chain Rule

\[ \frac{\partial}{\partial x} f_x(x, g(x)) = f_{xx}(x, g(x)) + f_{xy}(x, g(x)) g'(x) \]

and

\[ \frac{\partial}{\partial x} f_y(x, g(x)) g'(x) = (f_{yx}(x, g(x)) + f_{yy}(x, g(x)) g'(x)) g'(x) + f_y(x, g(x)) g''(x) \]

If \( f \) is \( C^2 \) then \( f_{yx}(x, g(x)) = f_{xy}(x, g(x)) \), so combining the results gives 17.13.

(b) If \( f_x(x_0, y_0) = 0 \) then by the conclusion of Dini’s Theorem, \( g'(x_0) = f_{x} (x_0, y_0) \). Thus by 17.13, after plugging in \( g'(x_0) = 0 \),

\[ g''(x_0) = -f_{xx}(x_0, g(x_0)) / f_y(x_0, g(x_0)) < 0 \]

note the condition \( f_y(x_0, g(x_0)) f_{xx}(x_0, g(x_0)) > 0 \) can be read, “\( f_y \) and \( f_{xx} \) both have the same sign.”

Thus from basic 1 dimensional calculus, the function \( g \) is (strictly) concave down near \( x_0 \), so there exists \( \delta > 0 \) such that if \( |x - x_0| < \delta \), then \( g(x) < g(x_0) = y_0 \).

41. Exercises 1 & 5, Section 17.2. Here, “analyze” means the following: prove which variables can be locally written as functions of the rest, and find the derivatives of these functions.

**Answer:**

17.2 #1 If \( F \) denotes the function \( \mathbb{R}^3 \to \mathbb{R}^2 \) given by the left side of the equations, then

\[ DF_{(x,y,z)} = \begin{bmatrix} 6x(x^2 + y^2 + z^2)^2 - 1 & 6y(x^2 + y^2 + z^2)^2 & 6z(x^2 + y^2 + z^2)^2 + 1 \\ -2x \sin(x^2 + y^4) & -4y^3 \sin(x^2 + y^4) & e^z \end{bmatrix} \]

so

\[ DF_{(0,0,0)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \]

the only nonsingular minor of this matrix is \( \det \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = -1 \) which corresponds to the variables \( x, z \). It follows by the Implicit Function Theorem (notice \( F \) is \( C^1 \)) that there exists \( \delta > 0 \) and a \( C^1 \) function \( G : B_\delta(0) \to \mathbb{R}^2 \) such that

\[ F(x, y, z) = (0, 0) \iff (x, z) = G(y) \]

for all \( y \in B_\delta(0) \) and \( (x, z) \in B_\delta(0, 0) \).
Also we must have
\[
DG_0 = -\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

The Implicit function theorem is inconclusive on whether the other variables can be solved for.

17.2 # 5 If \( f : \mathbb{R}^3 \to \mathbb{R} \) denotes the left hand side of the equation, then \( f \) is \( C^1 \) and
\[
Df_{(x,y,z)} = \begin{bmatrix} 2xe^{x^2} - 4y^3 & 2y - 12xy^2 & 1 \end{bmatrix}
\]
so
\[
Df_{(0,0,0)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}
\]
Therefore by the Implicit function theorem, \( z \) is locally a function of \( x \) and \( y \). Specifically, there exists \( \delta > 0 \) and continuously differentiable function \( g : B_\delta(0,0) \to \mathbb{R} \) such that \((x,y,g(x,y)) \) is in the level set for all \((x,y) \in B_\delta(0,0)\), and further if \((x,y,z) \) is a point in the level set with \( (x,y) \in B_\delta(0,0) \) and \( z \in B_\delta(0) \), then necessarily \( z = g(x,y) \).

The Implicit function theorem is inconclusive on whether the other variables can be solved for, although by graphing the level set you can see they cannot be.

42. (a) Let \( f : \mathbb{R} \to \mathbb{R} \) satisfy \( f(1) = 0 \). What additional conditions on \( f \) will allow the equation
\[
2f(xy) = f(x) + f(y)
\]
to be solved for \( y \) on some open set containing \((1,1)\)? Prove these conditions guarantee the solution.

(b) Obtain the explicit solution for the choice \( f(t) = t^2 - 1 \).

Answer:

(a) Claim: The conditions \( f \) is \( C^1 \) and \( f'(1) \neq 0 \) suffice. Let \( F(x,y) = 2f(xy) - f(x) - f(y) \). Also \( F \) is a \( C^1 \) function \( \mathbb{R}^2 \to \mathbb{R} \) since \((x,y) \to xy \) is \( C^1 \), and \( f \) is \( C^1 \), etc. Then
\[
DF_{(x,y)} = \begin{bmatrix} 2yf'(xy) - f'(x) & 2xf'(xy) - f'(y) \end{bmatrix}
\]
So \( DF_{(1,1)} = \begin{bmatrix} f'(1) & f'(1) \end{bmatrix} \). In particular if \( f'(1) \neq 0 \) then by Dini’s Theorem, there exists \( \delta > 0 \) and a function \( g : (1-\delta,1+\delta) \to \mathbb{R} \) such that if \( |x-1| < \delta \) and \( |y-1| < \delta \) and \( F(x,y) = 0 \), then \( y = g(x) \).
Finally just notice that \( F(x,y) = 0 \) iff \( x \) and \( y \) satisfy the original equation. Thus our open set is the rectangle \((1-\delta,1+\delta) \times (1-\delta,1+\delta)\), and \( y \) can be solved for on this set.
Remark: Notice that if the equation were

\[ f(xy) = f(x) + f(y) \]

then the result would be false. In fact the function \( \log x : (0, \infty) \to \mathbb{R} \) satisfies the equation for all \( x, y \), therefore we couldn’t possibly solve for \( y \) in terms of \( x \).

(b) Just plug in \( f \) and solve for \( y \) to get

\[ y = \sqrt{\frac{x^2}{2x^2 - 1}} \]

which gives us the solution as long as \( |x| > \sqrt{2}/2 \). Notice that we need to choose the positive square root, because that’s the only one that satisfies the equation at \((1,1)\).

43. A block matrix is a matrix built out of smaller matrices. For example, if \( A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \) and \( D = \begin{bmatrix} 5 \\ 9 \end{bmatrix} \) we denote

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
4 & 2 & 8 \\
1 & 0 & 1 \\
2 & 7 & 5 \\
3 & 4 & 9
\end{bmatrix}
\]

Also, throughout let \( 0 \) denote the matrix all of whose entries are 0.

(a) Suppose that \( A, B, \) and \( D \) are arbitrary \( 2 \times 2 \) matrices. Prove by directly computing the determinant that

\[
\det \begin{bmatrix}
A & B \\
0 & D
\end{bmatrix} = \det A \cdot \det D
\]

Here \( 0 \) is a \( 2 \times 2 \) matrix.

(b) In general, let \( A \) be an \( n \times n \) matrix, \( B \) an \( n \times m \) matrix, and \( D \) an \( m \times m \) matrix. Prove by induction on \( n, m \) that

\[
\det \begin{bmatrix}
A & B \\
0 & D
\end{bmatrix} = \det A \cdot \det D
\]

Here \( 0 \) is an \( m \times n \) matrix. (Hint: Show you may assume \( n \geq m \). Then use complete induction on \( n \). Also, Cramer’s Rule.)