Calculus 140, section 2.4 One-Sided and Infinite Limits

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Examples A–D: Consider the following functions. Why is it problematic to try to evaluate \( \lim_{x \to 1} f(x) \) for them?

A) \( f(x) = \sqrt{x - 1} \)  
B) \( f(x) = \frac{|x - 1|}{x - 1} \)  
C) \( f(x) = \frac{1}{x - 1} \)  
D) \( f(x) = \ln(x - 1) \)

Definition 2.4: “Let \( f \) be a function defined at each point of some open interval \((c, a)\). A number \( L \) is the limit of \( f(x) \) as \( x \) approaches \( a \) from the left (or is the left-hand limit of \( f \) at \( a \)) if for every \( \varepsilon > 0 \) there is a number \( \delta > 0 \) such that

\[
\text{if } a - \delta < x < a, \text{ then } |f(x) - L| < \varepsilon.
\]

In this case we write \( \lim_{x \to a^-} f(x) = L \) and say that the left-hand limit of \( f \) at \( a \) exists.”

The notation \( \lim_{x \to a^-} f(x) \) is read “the limit of \( f(x) \) as \( x \) approaches \( a \) from the left”.

If we were to consider some open interval \((a, c)\) to the right of \( a \), we get the analogous right-hand limit of \( f \) at \( a \). If \( \lim_{x \to a^+} f(x) = L \) we say that the right-hand limit of \( f \) at \( a \) exists.”

The notation \( \lim_{x \to a^+} f(x) \) is read “the limit of \( f(x) \) as \( x \) approaches \( a \) from the right”.

How do these one-sided limits connect to the ordinary, or two-sided limits of section 2.2?

Theorem 2.5 (short version): If both one-sided limits exist and also \( \lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \), then \( \lim_{x \to a} f(x) \) exists, and

\[
\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).
\]

Good news: All of the properties given in Lecture 2.3 (sum rule, constant multiple rule, etc.) apply to one-sided limits!

As always, you should read through the more detailed explanations in the text, and look over the text’s worked-out Examples.
Example A: Given \( f(x) = \sqrt{x-1} \), evaluate \( \lim_{x \to 1^-} f(x) \), \( \lim_{x \to 1^+} f(x) \) and \( \lim_{x \to 1} f(x) \).

Example B: Given \( f(x) = \frac{|x-1|}{x-1} \), evaluate \( \lim_{x \to 1^-} f(x) \), \( \lim_{x \to 1^+} f(x) \) and \( \lim_{x \to 1} f(x) \).
Example B extended: Given \( f(x) = \begin{cases} 2x - 1 & \text{for } x < 1 \\ 1 & \text{for } x > 1 \end{cases} \), evaluate \( \lim_{x \to 1^-} f(x) \), \( \lim_{x \to 1^+} f(x) \) and \( \lim_{x \to 1} f(x) \).

Example C: Given \( f(x) = \frac{1}{x - 1} \), evaluate \( \lim_{x \to 1^-} f(x) \), \( \lim_{x \to 1^+} f(x) \) and \( \lim_{x \to 1} f(x) \).
Definition 2.6 (short version): If \( \lim_{x \to a^-} f(x) = \infty \), or \( \lim_{x \to a^+} f(x) = -\infty \), then “the vertical line \( x = a \) is called a vertical asymptote of the graph of \( f \), and we say that we say that \( f \) has an infinite … limit at \( a \).”

Example C extended: Given \( f(x) = \frac{x-1}{x^2-1} \), find all vertical asymptotes.

Example D: Given \( f(x) = \ln(x-1) \), evaluate \( \lim_{x \to 1^-} f(x) \), \( \lim_{x \to 1^+} f(x) \) and \( \lim_{x \to 1} f(x) \).
The text considers the graph of \( f(x) = \sqrt[3]{x} \). While it has no vertical asymptotes, something interesting occurs when we consider the slope of the tangent at \( x = 0 \).

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \frac{x^{\frac{1}{3}} - 0}{x - 0} = \frac{1}{x^{\frac{2}{3}}} = \infty
\]

Definition 2.7: “Suppose \( f \) is continuous at \( a \). If \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \infty \) or \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = -\infty \) then we say that the graph of \( f \) has a vertical tangent at \((a, f(a))\). In that case the vertical line \( x = a \) is called the line tangent to the graph of \( f \) at \( a \).”