Calculus 140, section 3.1 Derivatives
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We include section 3.1 with Chapter 2 on Exam 1 because it really is just a small extension of the topics of Chapter 2.

From sections 2.1 and 2.2, we have that the slope of a line tangent to a graph at a point where \( x = a \) is

\[
m_a = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.
\]

Then, substituting into the (I hope) familiar point-slope formula \( y - y_1 = m(x - x_1) \) we get that the equation of a line tangent to a graph at a point where \( x = a \) is

\[
y - f(a) = m_a (x-a) \quad \text{or} \quad y = f(a) + m_a (x-a).
\]

We’re now going to formalize this into the definition of “one of the two central concepts of calculus: the derivative.”

Definition 3.1: “Let \( a \) be a number in the domain of the function \( f \). If \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \) exists, we call this limit the derivative of \( f \) at \( a \), and denote it by \( f'(a) \), so that

\[
\frac{f(x) - f(a)}{x - a} \to f'(a).
\]

If this limit exists, we’ll use terminology such as “\( f \) has a derivative at \( a \)” and “\( f \) is differentiable at \( a \)”.

The (first) derivative of \( f \) has several notations that we will use on a regular basis: \( f', f'(x), y', \frac{dy}{dx}, \frac{d}{dx} [f(x)] \).

Others that you might see in other texts include \( \dot{u}, \ Df(x), \ D_x f \). (Note the dot over the \( u \).) [See Table 3.1 in the text.]

Back to lines tangent to a curve.

By definition, the slope of the line tangent to a curve at the point \((a, f(a))\) is \( f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \).

By extension, the equation of the line tangent to the curve at the point \((a, f(a))\) is

\[
y - f(a) = f'(a)(x-a) \quad \text{or equivalently} \quad y = f(a) + f'(a)(x-a).
\]

Example A: Find the equation of the line tangent to \( f(x) = x^2 - 4 \) at \( x = 1 \).
Theorem 3.2: “If $f$ is differentiable at $a$, then $f$ is continuous at $a$, that is \( \lim_{x \to a} f(x) = f(a) \).”

This follows from the definitions of differentiable and continuity. See the text’s proof for details.

IMPORTANT: This is a one-way conditional statement! While differentiable implies continuity, continuous does not imply differentiable. See the text’s Example 3 which looks at $f(x) = |x|$ at $x = 0$.

It would be labor-intensive (and impossible) to use the definition above to find the (first) derivative of a given function for every one of the infinite points in its domain.

Instead, we’re going to generalize the process to find a formula that can be used for any value $x$ in the domain of a given function.

Specifically, given a differentiable function $f$, the (first) derivative of $f$ is given by \( f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \).

Example A revisited: Given $f(x) = x^2 - 4$, find a formula for the (first) derivative of $f$, that is, for $f'(x)$. Then, use your formula to find $f'(a)$ for various values $a$.

Hint for homework: You may find the text’s Example 5 \( f(x) = \sqrt{x} \) useful when searching for a technique to use for homework questions.

One last note on applications (i.e. word problems). In applications questions, the first derivative of some functions takes on a very specific meaning, one of which the text explores in Example 2, and you will be asked to evaluate in homework exercises:

“velocity is the derivative of the position function: $v(t) = f'(t)$

 marginal cost is the derivative of the cost function: $m_C(x) = C'(x)$

 marginal revenue is the derivative of the revenue function: $m_R(x) = R'(x)$”.