Calculus 140, section 3.6 Implicit Differentiation
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All of the equations encountered so far have been functions, \( y = f(x) \): for example \( y = 45x^2 - x^3 \) and \( P(x) = \frac{80}{2 + 3e^{-10x}} \). This is an *explicit* statement of the function formula, and given an explicit function and a value for \( x \), the determination of the corresponding \( y \)-coordinate becomes a calculation. In addition, the determination of the slope of the curve at that value of \( x \) means using one of the derivative rules developed so far, then calculating.

Even when a function is not expressed explicitly, it is sometimes possible to solve for the explicit version:
\[
5x + 2y = 12 \quad \Rightarrow \quad y = f(x) = -\frac{5}{2}x + 6.
\]

However, not all equations involving \( x \) and \( y \) can easily be rearranged algebraically into an explicit version, and others cannot be written explicitly at all. We’ll call these *implicit* equations. (See Examples below).

It is sometimes possible to find a derivative \( \frac{dy}{dx} \) from an implicit equation. The process is called *implicit differentiation*.

Example A: Given the equation \( 5x^2 + 2y^2 = 53 \), a) Verify that the point \( (x, y) = (-3, 2) \) satisfies the equation. b) Use implicit differentiation to find \( \frac{dy}{dx} \). c) Find the equation of the tangent to the curve at \( (x, y) = (-3, 2) \).

*answers:* \(-\frac{5x}{2y} \cdot \frac{15}{4}x + \frac{53}{4}\)

Example B: Given the equation \( \ln(x - y) = xy \), a) Use implicit differentiation to find \( \frac{dy}{dx} \). b) Find the equation of the tangent to the curve at \( (x, y) = (1, 0) \).

*answers:* \( \frac{xy - y^2 - 1}{-1 - x^2 + xy} \cdot \frac{1}{2}x - \frac{1}{2} \)
Example C: Given the equation $x^2y^3 = 1$, a) Use implicit differentiation to find $\frac{d^2y}{dx^2}$. b) Solve for the explicit equation and find $\frac{d^2y}{dx^2}$. c) Show that the results from (a) and (b) are equal.  \textit{answer:}  $\frac{10y}{9x^2}$

Example D: Given the equation $2x + 3y = e^{\sin(xy)}$, find $\frac{dy}{dx}$.  \textit{answer:}  $\frac{ye^{\sin(xy)} \cos(xy) - 2}{3 - xe^{\sin(xy)} \cos(xy)}$

Example A revisited: Suppose that the equation $5x^2 + 2y^2 = 53$ represents the path of an oval racetrack. The position coordinates $x$ and $y$ would each be a function of time $t$. Use implicit differentiation to find $\frac{dy}{dt}$ in terms of $x$, $y$ and $\frac{dx}{dt}$. [We’re finding the $y$-component of velocity!]