Calculus 140, section 5.3 Special Properties of the Definite Integral
7 tidbits about integrals you didn’t know you needed to know

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Definition 5.6, part 1: “Let \( f \) be continuous on \([a, b] \). Then \( \int_a^a f(x) \, dx = 0 \).”

It’s fairly obvious why this must be true.

\[
\lim_{\|P\| \to 0} \sum_{k=1}^1 f(t_k) \Delta x_k = 0
\]

In geometric area terms, a line segment, which is infinitely thin, has a width, and therefore an area, equal to 0. Among other things this means that the areas under a curve on the intervals \([a, b]\), \([a, b]\), \((a, b)\) and \((a, b)\) are all mathematically equal.

Definition 5.6, part 2: “Let \( f \) be continuous on \([a, b] \). Then \( \int_b^a f(x) \, dx = -\int_a^b f(x) \, dx \).”

We’ve already encountered a similar concept when we looked at distance and velocity: positive is up or forward, and negative is down or backward. The “integral from \(b\) to \(a\)” is the “integral from \(a\) to \(b\)” in reverse gear.

5.2 Example B revisited: Evaluate \( \int_2^2 2x \, dx \) and \( \int_5^2 2x \, dx \).

Theorem 5.7, Rectangle Property: “For any numbers \(a, b,\) and \(c\), \( \int_a^b c \, dx = c(b-a) \).”

Proof:

5.2 Example A revisited: Evaluate \( \int_{10}^2 5 \, dx \).
Theorem 5.8, Addition Property: “Let \( f \) be continuous on an interval containing \( a, b, \) and \( c \). Then
\[
\int_a^b \! f(x) \, dx = \int_a^c \! f(x) \, dx + \int_c^b \! f(x) \, dx.
\]

The text has put the highly-technical proof in the Appendix.

Example C: Given the piecewise continuous function \( f(x) = \begin{cases} 
2x & x < 3 \\
5 & x \geq 3
\end{cases} \), evaluate \( \int_0^6 f(x) \, dx \).

“Piecewise continuous” means the function is composed of pieces, each of which is continuous.
(See the text for the detailed explanation.)

Theorem 5.9, Comparison Property: “Let \( f \) be continuous on \([a, b]\), and suppose \( m \leq f(x) \leq M \) for all \( x \) in \([a, b]\). Then \( m(b-a) \leq \int_a^b \! f(x) \, dx \leq M(b-a) \).”

The value \( m(b-a) \) is a lower bound for the integral; \( M(b-a) \) is an upper bound for the integral.
The text’s proof, based on a Riemann sum, takes 5 lines of text and 3 lines of equations.

4.1 Example C revisited: Using the Comparison Property, find lower and upper bounds for \( \int_{-1}^4 x^3 \, dx \).

Corollary 5.10: “Let \( f \) be nonnegative and continuous on \([a, b]\). Then \( \int_a^b f(x) \, dx \geq 0 \).”

Proof:
Now we come to the 7\textsuperscript{th} and final tidbit of this section.

Theorem 5.11, Mean Value Theorem for Integrals: “Let $f$ be continuous on $[a, b]$. Then there is a number $c$ in $[a, b]$ such that $\int_{a}^{b} f(x) \, dx = f(c)(b-a)$.”

Proof:

The value $\frac{\int_{a}^{b} f(x) \, dx}{b-a} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx$ is called the \textbf{mean value} or \textbf{average value} of $f$ on $[a, b]$.

5.2 Example B once again: Find the mean value of $f(x) = 2x$ on $[2, 5]$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graph.png}
\caption{Graph of $f(x) = 2x$ on $[2, 5]$.}
\end{figure}

interpretation in calculus terms:

interpretation in geometric terms: