Calculus 141, section 7.8 Differential Equations (Methods)

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Section 7.7 introduced differential equations (DEs). Section 7.8 provides two methods for solving DEs. The first type is a separable differential equation, i.e. one in which we can separate elements containing the $x$ variable from ones containing the $y$ variable.

Separable DEs will generally have one of two forms:

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{dy}{g(y)} = f(x) \, dx$$

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \Rightarrow g(y) \, dy = f(x) \, dx$$

Example A: The temperature $y$ of a cup of coffee changes with time (Newton’s Law of Cooling). Find an equation to express temperature as a function of time. Answer: $y = Ke^{-0.1t} + 70$

We can easily check this general solution using the methods of section 7.7. Note several things:

1) The coefficient $-0.1$ is a “constant of proportionality” specific to this particular coffee cup, and would have been found via observation.

2) Domain of $\ln (y - 70)$, normally a concern, is not a worry in this case since the temperature of the coffee will not go below room temperature. Indeed, our equation for $y$ has a horizontal asymptote at $y = 70$.

3) While each of the two integrals would usually be written with a “$+ C$”, since these are arbitrary constants the two can be combined on one side of the equation.

4) We could leave the equation with $e^C$ as the coefficient, but again since $C$ is arbitrary, $e^C$ will itself be a constant and has been written more simply as $K$.

Example A extended: If a cup of coffee in a $70^\circ$ room began at $190^\circ$, what is its temperature after 5 minutes? Answer: $y(5) = 120e^{-0.5} + 70 \approx 142.78^\circ$ F

Example B: Solve $y' = \frac{3t^2}{y^2}$. Answer: $y = \sqrt[3]{3t^3} + C$
We already checked this result—see lecture notes 7.7 Example E—and found the particular solution for which \( y' = \frac{1}{3} \). Note that since \( C_1 \) is an arbitrary constant, we easily and legally replaced \( 3 \times C_1 \) with a more generic \( C \).

This method may seem pretty cavalier—we’re almost treating \( \frac{dy}{dt} \) as if it is a fraction, which of course it is not. Is our procedure legal? Of course, and we can justify it using substitution. When we have \( f(y) \frac{dy}{dt} = g(t) \), we take the integral of both sides with respect to \( t \). Letting \( u = y \), then \( du = \frac{dy}{dt} \cdot dt \), and thus

\[
\int f(y) \frac{dy}{dt} \, dt = \int f(u) \, du = \int f(y) \, dy.
\]

Example C: Solve \( y' - 2ty = t \). \textit{Answer:} \( y = Ce^{t^2} - \frac{1}{2} \)

Once again a complicated constant was replaced with the more generic \( C \). Also note that the particular solution we checked in section 7.7 lecture notes Example C fit this solution.

Example D: Solve \( y' = (1 + y^2)e^x \). \textit{Answer:} \( y = \tan(e^x + C) \)

Note how, while substitution helped solve the integral in Example C, it would not have worked here. We had to recognize \( \tan^{-1} \).
We now turn to a more difficult scenario: linear first-order differential equations. These are of the form

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

To solve these, we need to do some clever things. First we’ll define $S(x) = \int P(x) \, dx$, i.e. an anti-derivative.

Next, multiply both sides of the DE by $e^{S(x)}$ and integrate with respect to $x$.

$$e^{S(x)} \frac{dy}{dx} + P(x) \cdot e^{S(x)} \cdot y = e^{S(x)} \cdot Q(x)$$

$$\int e^{S(x)} \frac{dy}{dx} \, dx + \int P(x) \cdot e^{S(x)} \cdot y \, dx = \int e^{S(x)} \cdot Q(x) \, dx$$

$$\int f \, g' + f' \, g \, dx =$$

$$f' \cdot g = e^{S(x)} \cdot y = \int e^{S(x)} \cdot Q(x) \, dx$$

$$y = e^{-S(x)} \int e^{S(x)} \cdot Q(x) \, dx$$

In using this formula, it will be important to correctly identify $P$ and $Q$, and also to correctly construct the integrating factor $e^{S(x)}$. It is called an integrating factor because it transforms the left-hand side into a form that can be integrated because it is the result of differentiating by the Product Rule.

Example E: Solve $xy' = x^2 + 3y$, $x > 0$. *Answer:* $y = -x^2 + Cx^3$, $x > 0$

Note that $S$ can be any anti-derivative, so we might as well pick the simplest one and make our life a little easier.

Also note, that instead of using a memorized formula $y = e^{-S(x)} \int e^{S(x)} Q(x) \, dx$, I wrote the left-hand side as the antiderivative of a product of functions. Feel free to use whichever method works for you.

Example E extended: Given the initial condition $y(1) = 2$, find the particular solution. *Answer:* $y = 3x^3 - x^2$
Example F: An anti-coagulant is introduced intravenously at a rate of 0.5 mg per hour. At the same time, 2% of the drug in the bloodstream is absorbed into the body. Solve $y' = -0.02y + 0.5$ to get an equation describing the amount of anti-coagulant in the bloodstream. \textit{Answer:} $25 + Ce^{-0.02t}$

Examples G: $xy' = x^2 + y^2$ is not linear, and $xy'' = x^2 + y$ is not first-order, so we cannot use the integrating factor method for either of these.

Bonus Example H: Solve $y' - y^3 \cos t = 0$.
This one is easily separable.
\[
y' = y^3 \cos t \quad \Rightarrow \quad y^{-3} \frac{dy}{dt} = \cos t \quad \Rightarrow \quad \int y^{-3} \, dy = \int \cos t \, dt \quad \Rightarrow \quad -\frac{1}{2}y^{-2} = \sin t \quad \Rightarrow \quad -\frac{1}{2\sin t} + C = y^2
\]
\[
\Rightarrow \quad y = \pm \sqrt{\frac{1}{2\sin t} + C}
\]

Notes: The general solution is actually two functions, the “plus” and “minus” versions. The correct one would be determined by initial conditions, when available.

Bonus Example I: Solve $y' + 2ty = 6t$.
This is already in the standard form for a linear first-order DE.
\[P(t) = 2t \text{ and } Q(t) = 6t. \text{ Then } S(t) = \int 2t \, dt = t^2 \text{ and the integrating factor is } e^{t^2}.
\]
\[
ed^{t^2} y' + 2te^{t^2} y = 6te^{t^2} \quad \Rightarrow \quad \int \frac{d}{dt} \left[ e^{t^2} y \right] \, dt = \int 6te^{t^2} \, dt \quad \Rightarrow \quad u = t^2, \, du = 2t \, dt \quad \Rightarrow \quad e^{t^2} y = 3\int e^{u} \, du
\]
\[
\Rightarrow \quad e^{t^2} y = 3e^{t^2} + C \quad \Rightarrow \quad y = 3 + Ce^{-t^2}
\]