A sequence \( \{a_n\}_{n=m}^{\infty} \) consists of an ordered set of numbers. If we were to begin adding the numbers of a sequence together, \( s_j = a_m + a_{m+1} + \ldots + a_{m+j-1} \), we would have a partial sum, designated as \( \sum_{n=m}^{m+j-1} a_n \). The sum \( s_j \) above of the first \( j \) terms is called the \( j \)th partial sum. If we take the sum to infinity, \( \sum_{n=m}^{\infty} a_n \), then we have an infinite series, that is, an infinite sum. But what does it mean to add an infinite sequence of numbers, and can such an infinite series add up to a finite number? In other words, does an infinite series have a limit, does it converge?

**Definition:** Given a sequence \( \{a_n\}_{n=m}^{\infty} \) and a series \( \sum_{n=m}^{\infty} a_n \), if \( \lim_{j \to \infty} (a_m + a_{m+1} + \ldots + a_{m+j-1}) \) exists, then \( \sum_{n=m}^{\infty} a_n = \lim_{j \to \infty} (a_m + a_{m+1} + \ldots + a_{m+j-1}) \) and the series converges. If the limit goes to \( \infty \) or does not exist, then the series diverges.

How can we determine whether or not a series converges? One way to approach this is to remember that we have two sequences involved in every series. The first is the sequence of terms \( \{a_n\}_{n=m}^{\infty} \). The second is the sequence of partial sums: \( s_m = a_m \), \( s_{m+1} = a_m + a_{m+1} \), \( s_{m+2} = a_m + a_{m+1} + a_{m+2} \), \ldots . If we can show that the sequence of partial sums \( \{s_j\} \) converges, we will be able to conclude that the series converges.

**Example A:** Does the series \( \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \) converge, and if so, to what limit?

**Intuitive Answer:** converges to what value?

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Be careful that you are clear in your own mind about the differences among the sequence, the sequence of partial sums, and the series.
sequence
\[ \{a_n\} = \left\{ \frac{1}{2^n} \right\}_{n=0}^{\infty} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \]

sequence of partial sums
\[ \{s_j\} = \left\{ 2 - \frac{1}{2^j} \right\}_{j=0}^{\infty} = 2 - 1, 2 - \frac{1}{2}, 2 - \frac{1}{4}, 2 - \frac{1}{8}, 2 - \frac{1}{16}, \ldots \]

series
\[ \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \]

Example A extended: Find \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) and \( \sum_{n=2}^{\infty} \frac{1}{2^n} \). \textit{Answers:} converges to 1; converges to \( \frac{1}{2} \)

Example A extended again: Find \( \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \) and \( \sum_{m=0}^{\infty} \frac{1}{2^{m+2}} \). \textit{Answers:} converges to 1; to \( \frac{1}{2} \)

Example B: Determine whether or not the series \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 7n + 12} \) converges and if so, find its limit.
Example C: Determine whether \( \sum_{n=1}^{\infty} (-1)^n \) converges, and if so, find its limit. \textit{Answer:} diverges

The text proves a couple of theorems that formalize the relationship between a series and its related sequence. Theorem 9.8 states: If \( \sum_{n=m}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \). The contrapositive, which is logically equivalent is stated as Corollary 9.9: If \( \lim_{n \to \infty} a_n \neq 0 \) or does not exist, then \( \sum_{n=m}^{\infty} a_n \) diverges.
Note that in Examples A and B above (convergent series), \( \lim_{n \to \infty} \frac{1}{2^n} = 0 \) and \( \lim_{n \to \infty} \frac{1}{n^2 + 7n + 12} = 0 \), while in Example C (divergent series), \( \lim_{n \to \infty} (-1)^n \) does not exist. Also, from Lecture 9.3 Example A in, we now know that \( \sum_{n=1}^{\infty} \sqrt{n^2 + 3n - n} \) diverges since \( \lim_{n \to \infty} \sqrt{n^2 + 3n - n} = \frac{3}{2} \neq 0 \).

**Important cautionary note** (again): Be sure that you are clear in your own mind that there is a difference between the sequence of terms for \( \{a_n\}_{n=m}^{\infty} \) and the sequence of partial sums \( \{s_j\}_{n=m}^{\infty} \) produced by the series \( \sum_{n=m}^{\infty} a_n \).

The converse of Theorem 9.8 is **not** true: \( \lim_{n \to \infty} a_n = 0 \) is not a guarantee that \( \sum_{n=m}^{\infty} a_n \) will converge.

Although the harmonic sequence \( \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \) converges to 0, the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges (see text Example 4). Thus (Lecture 9.3 Example B) since the sequence \( \left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty} \) converges to 0, although we can say that the series \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) **might** converge, we cannot be certain that it does converge.

The text also proves some very convenient theorems which ease the task of evaluating series. Theorem 9.10 states: For \( c \neq 0 \) and \( m \geq 0 \), the geometric series \( \sum_{n=m}^{\infty} c r^n \) converges if and only if \( |r| < 1 \), and if it converges we can calculate the sum of the series: \( \sum_{n=m}^{\infty} c r^n = \frac{c r^m}{1-r} \).

The proof relies on calculating the sequence of partial sums and identifying the pattern which emerges.

Example A, \( \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \) is a geometric series with \( c = 1 \) and \( r = \frac{1}{2} \), and

\[
\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n = \frac{(\frac{1}{2})^0}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2 \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{2^n} = \sum_{n=2}^{\infty} \left( \frac{1}{2} \right)^n = \frac{(\frac{1}{2})^2}{1-\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.
\]

Theorem 9.11 and associated theorems give us means of combining series in much the same way that we can add (and subtract) and multiply (and divide) limits:

a. If \( \sum_{n=m}^{\infty} a_n \) and \( \sum_{n=m}^{\infty} b_n \) both converge, then \( \sum_{n=m}^{\infty} (a_n + b_n) \) converges and \( \sum_{n=m}^{\infty} (a_n + b_n) = \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n \).

b. For any number \( c \), if \( \sum_{n=m}^{\infty} a_n \) converges, then \( \sum_{n=m}^{\infty} c a_n \) also converges and \( \sum_{n=m}^{\infty} c a_n = c \sum_{n=m}^{\infty} a_n \).
Example D: Evaluate \( \sum_{n=0}^{\infty} \frac{4 + 2^n}{3^{n+1}} \). Answer: converges to 3

Method: Use factoring and separation of fractions to rearrange the series into the form of a geometric series and use the formula from Theorem 9.10 to evaluate: \( \sum_{n=m}^{\infty} c r^n = \frac{c r^m}{1-r} \) with \( |r| < 1 \).

\[
\text{factor: } \sum_{n=0}^{\infty} \frac{4 + 2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{4 + 2^n}{3^n}
\]

\[
\text{separate the fractions (distribute the division): } = \frac{1}{3} \sum_{n=0}^{\infty} \frac{4}{3^n} + \frac{1}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n}
\]

\[
\text{rewrite as } c r^n: \quad = \frac{4}{3} \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n + \frac{1}{3} \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^n
\]

\[
\text{apply Theorem 9.10: } \quad = \frac{\frac{4}{3} \left( \frac{1}{3} \right)^0}{1 - \frac{1}{3}} + \frac{\frac{1}{3} \left( \frac{2}{3} \right)^0}{1 - \frac{2}{3}}
\]

\[
\text{simplify: } \quad = \frac{\frac{4}{3} \cdot 1}{\frac{2}{3}} + \frac{\frac{1}{3} \cdot 1}{1 - \frac{2}{3}} = 2 + 1 = 3
\]